On the growth of the Kronecker coefficients

New stability properties through factorization of generating series.

Emmanuel Briand †, Amarpreet Rattan‡ and Mercedes Rosas.†

(†) Universidad de Sevilla (‡) Birkbeck, University of London.

ebriand@us.es, a.rattan@bbk.ac.uk. mrosas@us.es

(†) Supported by MTM2016-75024-P, P12-FQM-2696 , FQM333 and FEDER.

Abstract

By means of factorizations of Schur generating series, we discover two properties related to stability of the Kronecker coefficients:

- *hook stability*: Kronecker coefficients stabilize when the first row and first column of their three indexing partitions are increased in a balanced way.
- *linear growth*: Kronecker coefficients grow linearly when adding repeatedly three balanced 2–parts partitions with long enough first row to the indexing partions.

Schur generating series are obtained for the limits (hook stability) and dominant coefficient (linear growth)

Background: Stability of Kronecker coefficients.

- The Kronecker coefficients $g_{\lambda,\mu,\nu}$ are a family of integers indexed by triples of integer partitions of some integer n.
- They are the multiplicities of irreducible representations in the tensor products of irreducible complex representations of the symmetric groups (the Specht modules $\mathbb{S}_\lambda$):

$$\mathbb{S}_\mu \otimes \mathbb{S}_\nu \cong \bigoplus g_{\lambda,\mu,\nu} \mathbb{S}_\lambda.$$

- In terms of symmetric functions, the Kronecker coefficients are commonly defined as

$$s_\mu * s_\nu = \sum_\lambda g_{\lambda,\mu,\nu} s_\lambda,$$

where s_λ is the Schur function and $*$ is the *Kronecker product* defined by

$$p_\lambda * p_\mu = \delta_{\lambda,\mu} z_\lambda^{-1} p_\lambda.$$

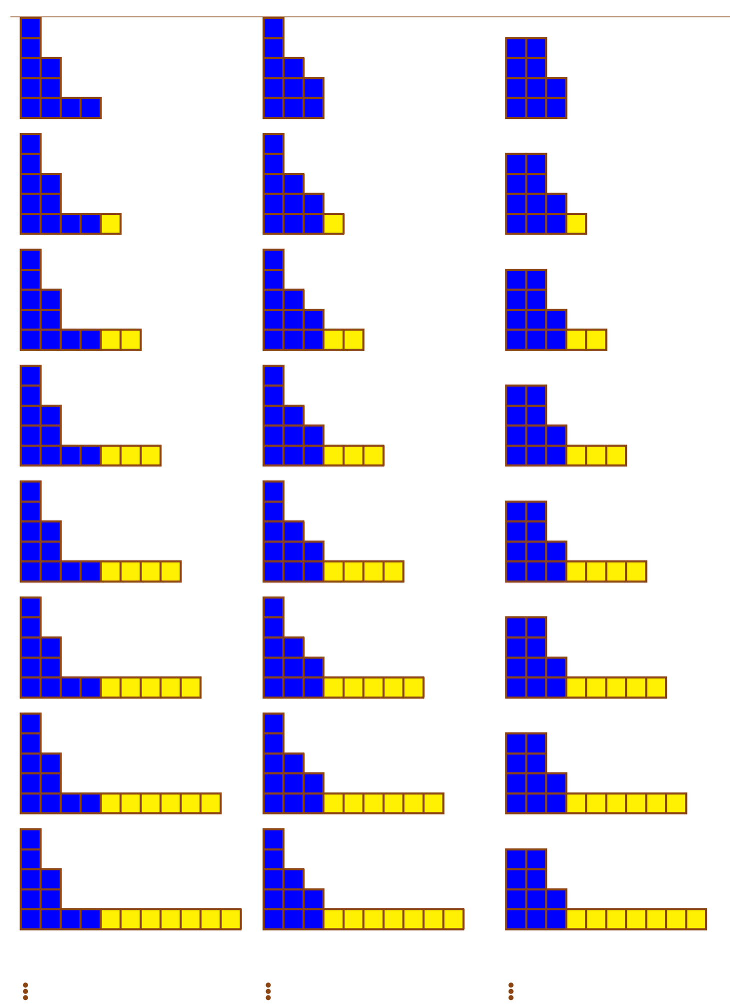
- The classic stability property of Murnaghan [2] in 1938: the sequences of Kronecker coefficients obtained by incrementing simultaneously and repeatedly the first part of three partitions λ, μ and ν, are all weakly increasing and eventually constant.

EXAMPLE: The sequence of Kronecker coefficients

$$g_{(4+n,2,2,1,1),(3+n,3,2,1,1),(3+n,3,2,2)}, n = 0, 1, 2, \ldots$$

takes the following values.

λ μ ν $g_{\lambda,\mu,\nu}$



$g_{\lambda,\mu,\nu}$
17
17
119
256
305
308
308
308
308

The stable value is 308. This stable value is called the *stable* or *reduced Kronecker coefficient* $\overline{g}_{(2,2,1,1),(3,2,1,1),(3,2,2)}$. It is indexed by the partitions obtained by removing the first part in the triples of partitions in the sequence. For stable Kronecker coefficients, the indexing partitions need not be partitions of the same integer.

- Murnaghan's stability property has since received several different proofs from different mathematical areas, including geometric, algebraic and representation–theoretic proofs. Recently a much more general property has been established: if $g_{n\alpha,n\beta,n\gamma} = 1$ for all $n > 0$, then all sequences of Kronecker coefficients

$$g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma}, n = 0, 1, 2 \ldots$$

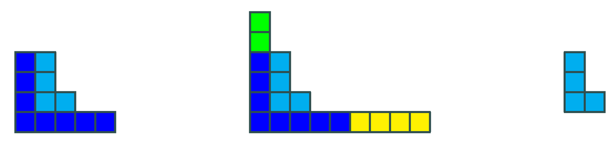
are weakly increasing and eventually constant [3].

- The asymptotic properties that we consider in the present work are of a different kind.

Hook stability.

We consider what happens when we increment simultaneously the first row and first column of the three partitions indexing a Kronecker coefficient. Let $\lambda \oplus (i|j)$ be the partition obtained from λ by adding to its diagram i boxes in the first row and j boxes in the first column. Also, denote by $\widehat{\lambda}$ the partition obtained from λ by removing the first row and first column.

λ $\lambda \oplus (4|2)$ $\widehat{\lambda}$



Our first result is

Theorem 1. *For any triple of non–empty partitions λ, μ and ν of the same weight, there exists a constant $\overline{\overline{g}}_{\widehat{\lambda},\widehat{\mu},\widehat{\nu}}$ that only depends on the partitions $\widehat{\lambda}$, $\widehat{\mu}$ and $\widehat{\nu}$, such that: for all $(a, b, c, m) \in \mathbb{N}^4$ with $m \geq a, b, c$, and $m - (a+b+c)/2$, $a+b-c$, $a+c-b$ and $b+c-a$ big enough, we have*

$$g_{\lambda\oplus(m-a|a),\mu\oplus(m-b|b),\nu\oplus(m-c|c)} = \overline{\overline{g}}_{\widehat{\lambda},\widehat{\mu},\widehat{\nu}}. \quad (1)$$

EXAMPLE: Consider the Kronecker coefficients $g_{\lambda\oplus(i|j),\lambda\oplus(i|j),\lambda\oplus(i|j)}$ for $\lambda = (3,3)$ and i and j from 0 to 9.

i \ j	0	1	2	3	4	5	6	7	8	9
0	0	1	5	5	1	0	0	0	0	0
1	1	8	27	40	30	11	1	0	0	0
2	1	15	53	89	91	64	33	11	1	0
3	2	19	62	108	129	122	97	64	33	11
4	2	19	63	112	138	141	135	122	97	64
5	2	19	63	112	139	145	144	141	135	122
6	2	19	63	112	139	145	145	145	144	141
7	2	19	63	112	139	145	145	145	145	145
8	2	19	63	112	139	145	145	145	145	145
9	2	19	63	112	139	145	145	145	145	145

- Each column stabilizes: this is Murnaghan's stability.
- Hook stability shows up as the grey region where all values are 145. This is the value of $\overline{\overline{g}}_{(2,2),(2,2),(2,2)}$.

Linear Growth.

Our second result is not a result of stability but still describes the asymptotic properties of some sequences of Kronecker coefficients.

Theorem 2. *Let (λ,μ,ν) and (α,β,γ) be two triples of partitions with α, β and γ having at most two parts.*
Assume that

- *all six partitions have their first part big enough (this corresponds to explicit linear inequalities).*
- *$(\alpha_2, \beta_2, \gamma_2)$ fulfil all three strict triangular inequalities*

$$\beta_2 + \gamma_2 > \alpha_2, \quad \alpha_2 + \gamma_2 > \beta_2, \quad \alpha_2 + \beta_2 > \gamma_2.$$

- *there exists n such that $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma}$ is non–zero.*

Then for $n \to \infty$,

$$g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma} \sim \frac{1}{2} A_{\overline{\overline{\lambda}},\overline{\overline{\mu}},\overline{\overline{\nu}}} \cdot m \cdot n,$$

where

- *$m = \max(\beta_2 + \gamma_2 - \alpha_2, \alpha_2 + \gamma_2 - \beta_2, \alpha_2 + \beta_2 - \gamma_2)$;*
- *$A_{\overline{\overline{\lambda}},\overline{\overline{\mu}},\overline{\overline{\nu}}}$ is a non-zero integer depending only on the partitions $\overline{\overline{\lambda}}, \overline{\overline{\mu}}, \overline{\overline{\nu}}$ obtained from λ, μ and ν by removing their first two parts.*

Methods: generating functions.

- Given any triple of partitions λ, μ and ν, we consider
  1) the stable Kronecker coefficients $\overline{g}_{\lambda\cup(1^a),\mu\cup(1^b),\nu\cup(1^c)}$.
  2) the stable Kronecker coefficients $\overline{g}_{\lambda+(a),\mu+(b),\nu+(c)}$.

  Case 1 corresponds to hook stability for (ordinary) Kronecker coefficients, and Case 2 to linear growth of (ordinary) Kronecker coefficients.
- We obtain for each of these two families a compact expression for its generating series. For this we use the *vertex operator* Γ for symmetric functions (as in [4]) where

$$\Gamma : s_\lambda \to \sum_n s_{(n,\lambda)} t^n;$$

that is, the operator Γ sends the symmetric function s_λ to the formal series $\sum_n s_{(n,\lambda)} t^n$. The latter can be expressed in the λ–ring formalism as

$$\sigma[tX] s_\lambda[X - 1/t],$$

where $\sigma[tX] = \sum_{k=0}^{\infty} h_k[X] t^k$, the generating series of complete sum symmetric functions.
- Both generating series factor as

Series × Polynomial.

More precisely, the series on the left–hand side of the factorization are respectively:

1) the series of the stable Kronecker coefficients $\overline{g}_{(1^a),(1^b),(1^c)}$, which are 1 when a, b and c fulfil the triangular inequalities, and 0 otherwise.
2) the series of the stable Kronecker coefficients $\overline{g}_{(a),(b),(c)}$, which depend piecewise-linearly on a, b, c when a, b and c fulfil the triangular inequalities, and are 0 otherwise.

- These series determine in each case the growth order of the corresponding family of stable Kronecker coefficients (stability/ linear growth). The polynomial on the right–hand side determines the limit value $\overline{g}$ (case of hook stability) or term A (case of linear growth).
- The stable Kronecker coefficients indexed by partitions with long first column (resp. long first row) are the Kronecker coefficients indexed by partitions with long first row and long first column (resp. long two first rows).

Schur generating series.

- The *Schur generating series* for a family of constants $C_{\lambda,\mu,\nu}$ indexed by triples of partitions is

$$\sum_{\lambda,\mu,\nu} C_{\lambda,\mu,\nu} s_\lambda[X] s_\mu[Y] s_\nu[Z],$$

where X, Y, and Z are independent alphabets, and the s_λ are the Schur functions.
- Many interesting families of constants have Schur generating series that admit a compact form involving the generating series σ of the complete sum symmetric functions h_k and operations on alphabets (λ–ring formalism).
- Our methods provide naturally such generating series for the coefficients $\overline{\overline{g}}_{\lambda,\mu,\nu}$ and $A_{\lambda,\mu,\nu}$. They appear in the last two lines in the following table.

Schur generating series	Coefficients
$\sigma[XY + XZ]$	Littlewood–Richardson coefficients $c_{\lambda,\mu,\nu}$
$\sigma[XYZ]$	Kronecker coefficients $g_{\lambda,\mu,\nu}$
$\sigma[XYZ + XY + XZ + YZ]$	Stable Kronecker coefficients (limits in Murnaghan's stability) $\overline{g}_{\lambda,\mu,\nu}$
$\sigma[XYZ + (1-\varepsilon)W)]$	Limits in hook stability $\overline{\overline{g}}_{\lambda,\mu,\nu}$.
$\sigma[XYZ + 2W]$	Coefficients $A_{\lambda,\mu,\nu}$.

Notations in this table:

- $W = XY + XZ + YZ + X + Y + Z$.
- The alphabet $-\varepsilon$ exchanges complete sums and elementary functions: $h_k[-\varepsilon X] = e_k[X]$.

References

[1] E. Briand, A. Rattan, and M. Rosas. On the growth of the Kronecker coefficients. *ArXiv e-prints*, July 2016.

[2] F. D. Murnaghan. The Analysis of the Kronecker Product of Irreducible Representations of the Symmetric Group. *Amer. J. Math.*, 60(3):761–784, 1938.

[3] S. V. Sam and A. Snowden. Proof of Stembridge's conjecture on stability of Kronecker coefficients. *Journal of Algebraic Combinatorics*, 43(1):1–10, 2015.

[4] J-Y. Thibon. Hopf algebras of symmetric functions and tensor products of symmetric group representations. *Internat. J. Algebra Comput.*, 1(2):207–221, 1991.