DUALITY FOR COUPLES OF CONICS

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Abstract. Consider the couples of distinct proper non-empty real projective conics. A rigid isotopy for such an object is a continuous deformation of the defining equations of the two conics, not modifying the singularity of the intersection points. Each class of rigid isotopy corresponds to a "configuration" of a couple of conics.

In the present paper, we show that if two couples of conics are in the same configuration, so are the corresponding couples of tangential conics. We make explicit the induced bijection between the configurations of in the primal space and the configurations in the dual space.

Keywords: pairs of conics, duality, rigid isotopy.

MSC2000: 15A63, 11E10, 68U05.

Contents

1. introduction

We consider here real projective conics: the zero loci of real ternary quadratic forms in $\mathbb{CP}^2$. They can be identified with the real ternary quadratic forms up to proportionality.

A *proper* conic is the zero locus of a *non–degenerate* quadratic form.

The goal of this paper is to answer to the following question: suppose you know that some couple of conics is in a given configuration. Does this determine the configuration of the associated tangential conics ? The answer is affirmative, and the correspondence is made explicit in section 5. Before, we start with making precise what we mean by "configuration" (this is elucidated by means of a notion of rigid isotopy,

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Orbit	I	Ia	Ib	II	IIa	III	IIIa	IV	V
real points	1111	–	11	211	2	22	–	31	4
imaginary points	–	1111	11	–	11	–	22	–	–

TABLE 1. The names of the nine strata. The strata are characterized by the multiplicities of the real and imaginary base points of each couple belonging to it.

section 2) and by some reminders about duality for conics (section 3). We finish we an example of application (section 6).

2. Rigid isotopy for couples of proper real conics

The intersection of the two conics is called their *base*. It is, generically, a set of four points of $\mathbb{CP}^2$. In the singular cases, some intersection points are multiple, but always the intersection is a finite set of points of $\mathbb{CP}^2$, whose multiplicities add up to 4.

Classify the points obtained as an intersection of two real algebraic curves as follows: say they are equivalent if there exist a local real analytic isomorphism that sends one to the other. Then the points obtained as an intersection of two distinct proper conics are totally classified by their multiplicity (1, 2, 3 or 4) and their nature (real or imaginary).

Let CQ_2 be the space of couples of distinct proper real projective conics with non-empty sets of real points. Classify the elements of CQ_2 according to the nature of their base, that is the number of real and imaginary points of each multiplicity. This decomposes CQ_2 into nine strata. See table 1 for the nomenclature used (taken from [4]). A *rigid isotopy class* for a couple of distinct proper conics is a connected component of some stratum. If two elements of CQ_2, call them (C_1, C_2) and (D_1, D_2), are in the same class, one says they are *rigid isotopic*. This means that there exists some continuous deformation of the equations of (C_1, C_2) which transforms them into the equations of (D_1, D_2), and preserves at each moment the singularity of each point in the base. This transformation itself – a continuous path in one class – is called a rigid isotopy.

It was shown in [3] that if (C_1, C_2) and (D_1, D_2) are rigid isotopic, there also exists an isotopy of the whole real projective plane $\mathbb{RP}^2$ transforming $\mathbb{R}(C_1)$ and $\mathbb{R}(C_2)$, the sets of real points of C_1 and C_2, into $\mathbb{R}(D_1)$ and $\mathbb{R}(D_2)$ respectively. Thus one can say that two couples of conics that are rigid isotopic are *in the same configuration*.

The paper [3] also showed that there are 20 rigid isotopy classes. A representant of each of the classes is drawn in Figure 1. There are two kinds of classes. The *symmetric classes* have the following property:

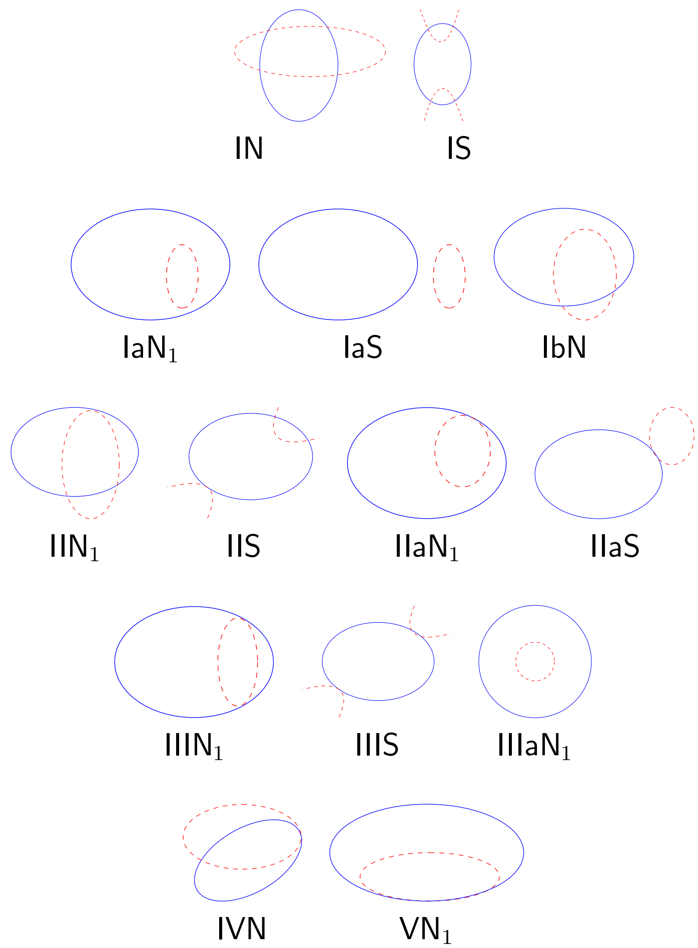

FIGURE 1. The rigid isotopy classes.

each time the couple (C_1, C_2) is in the class, so is the couple (C_2, C_1). The other classes, the *non–symmetric* ones, go by pairs. Each time a couple (C_1, C_2) is in one class of the pair, the couple (C_2, C_1) is in the other class. Moreover, one of the class of the pair is characterized by the fact that $\mathbb{R}(C_1)$ lies inside[1] $\mathbb{R}(C_2)$, and the other class by the fact that $\mathbb{R}(C_2)$ lies inside $\mathbb{R}(C_1)$.

The nomenclature is the following: the name of the rigid isotopy class of a couple of conics starts with the name of the orbit of the pencil it generates (as indicated in table 1), followed by a letter N or S (whose meaning is explained in [3]). If the class is non-symmetric, an index 1 or 2 is appended to indicate either that the first conic lies inside the second, or the reciprocal.

[1]The real locus of a real proper conic cuts the real projective plane into two connected components: the *inside*, homeomorphic to a disk, and the *outside*, homeomorphic to a Möbius strip.

Figure 1 doesn't display the non-symmetric classes with index 2, since they are obtained trivially from the same class with index 1.

3. Duality and conics

Any quadratic form f on a vector space V has a *tangential quadratic form*: this is the quadratic $\widetilde{f}$ form on the dual V^* of V that can be defined as follows: if F is the symmetric matrix representing f in some base, then the symmetric matrix representing $\tilde{f}$ in the dual base is the matrix of the cofactors of F. This polynomial mapping induces a mapping

$$C \mapsto C^\vee$$

from the space of real projective conics of $\mathbb{P}^2$ to the space of real projective conics of the dual projective space $\mathbb{P}^{2*}$. It is well-known that the restriction of this latter to *proper* conics drawn in $\mathbb{P}^2$ induces a bijection to the space of proper conics of $\mathbb{P}^{2*}$. Moreover, its reciprocal is the the similar bijection from $\mathbb{P}^{2*}$ to $\mathbb{P}^{2**} \cong \mathbb{P}^2$. That is, if C is proper, then $(C^\vee)^\vee = C$. Last, the points of $C^\vee$ represent lines of $\mathbb{P}^2$. If C is proper, then $C^\vee$ is the set of the tangents to C. We have also an interpretation for the inside and the outside of $C^\vee$. Let L be a real line of $\mathbb{P}^2$, considered as a point of $\mathbb{P}^*$. Consider the case when L, seen as a point of $\mathbb{P}^{2*}$, lies outside $C^\vee$. Then no real tangent to $C^\vee$ pass through L. Back into $\mathbb{P}^2$, this means that no real point of C lies on L. Similarly, L lies outside $C^\vee$ if and only if L cuts two times C.

See [2] for more details about duality for conics.

4. Rigid isotopy is preserved by duality

In this section, we check that the image of a rigid isotopy class for couples of real conics in $\mathbb{P}^2$ is a rigid isotopy class for couples of real conics in $\mathbb{P}^{2*}$. This means that the tangential map induces an involution on the set of rigid isotopy classes. This involution will be made explicit in section 5.

Call $\mathcal{T}$ the mapping $C \mapsto C^\vee$ of section 3, defined on the set of proper conics. We want to show that if θ is a rigid isotopy, then $\mathcal{T} \circ \theta$ is also a rigid isotopy. The continuity of $\mathcal{T} \circ \theta$ is trivial, since $\mathcal{T}$ is polynomial. The less trivial part consists in showing there is no change in the numbers of real and imaginary base points of each multiplicity.

We start with the following result.

Lemma 1. *Let C, D be two distinct proper conics. Let p be an intersection point of multiplicity $k > 1$. Let L be the common tangent at p for C and D.*

Then the tangential conics meet at L with multiplicity k and common tangent p.

Proof. Let $[x : y : z]$ be homogeneous coordinates on $\mathbb{CP}^2$. Let $[X : Y : Z]$ be the dual coordinates.

After convenient change of coordinates, the point p is $(0 : 0 : 1)$ and the tangent L is $[y = 0]$. Working in the affine chart $[z = 1]$, the two conics have local equations:

$$y = f(x), \quad y = g(x),$$

where f and g are some functions analytic at 0 of order 2. The multiplicity of intersection k is the smallest degree where the Taylor expansions of f and g differ.

In the dual space, this corresponds to the following situation: the dual conics meet at $(0 : 1 : 0)$ with common tangent $[Z = 0]$. In the affine chart $[Y = 1]$, the dual conics have parameterizations:

$$\begin{cases} X = & -f(x) + xf'(x) \\ Z = & -f'(x) \end{cases}, \qquad \begin{cases} X = & -g(x) + xg'(x) \\ Z = & -g'(x) \end{cases}$$

Since f has order 2, f' has order 1 and one can express x as an analytic function of Z of order 1:

$$x = F(Z).$$

Substituting $F(Z)$ for x in $X = -f(x) + xf'(x)$, one gets an analytic function $X = F_2(Z)$ of order 2. The coefficient of Z^i in its Taylor series only depends on the coefficients of degrees $\leq i$ in the Taylor series of f'.

Similarly for g.

As a consequence, the Taylor series expansions of F_2 and G_2 coincide up to order k, at least. Let K be the multiplicity of L as intersection of the dual curves, we have established that $K \geq k$.

Now the same construction can be done starting from the dual conics. This gives the reciprocal inequality: $K \leq k$, and the equality $K = k$ follows. □

Besides, if the conics C and D are real, and if the point p is real, then the tangent L is real. Thus, the numbers of real and imaginary points of each multiplicity > 1 are conserved. Because the number of simple points is 4 minus the sum of multiplicities of the multiple points, the global number of simple points (real and imaginary) is also conserved[2].

This is enough for the following theorem.

Theorem 1. *The tangential map sends a rigid isotopy to a rigid isotopy.*

Proof. As remarked before, the difficulty is to show that the numbers of real and imaginary base points of each multiplicity don't change along the image of the rigid isotopy.

[2]Nevertheless this doesn't imply that the number of *real* simple points is conserved. Actually this is not the case: remark for instance, in the next section, that Classes IIS and IIaS are exchanged under duality

class	real int.	imag. int.	real c.t.	imag. c.t.
IN_i	1111		1111	
IaN_i		1111		1111
IbN_i	11	11	11	11
IIN_i	211		211	
IIaN_i	2	11	2	11
IIIN_i	22		22	
IIIS	22		22	
IIIaN_i		22		22
IVN	31		31	
VN_i	4		4	
IS	1111			1111
IaS		1111	1111	
IIS	211		2	11
IIaS	2	11	211	

TABLE 2. multiplicities of the real and imaginary intersection points (*int.*) and common tangents (*c.t.*).

Because of the remark after the proof of lemma 1, one has only to check that the numbers of real and imaginary *simple* points don't change. But such a change would imply, at some point in the path, a coalescence, and thus a change into the numbers of real and imaginary multiple points. □

5. The involution induced by duality

As a consequence of the results of the previous section, the tangential map induces an involution on the set of rigid isotopy classes. By testing on representants, one gets the exact description of the involution (note this is also obvious on drawings, remarking that the intersection points of $C_1^\vee$ and $C_2^\vee$ correspond to the common tangents to C_1 and C_2).

Theorem 2. *The tangential map induces the following involution on the set of rigid isotopy classes:*

- *for every pair of classes of type N_i, the two members are exchanged: $\mathsf{IN}_1 \leftrightarrow \mathsf{IN}_2$, $\mathsf{IaN}_1 \leftrightarrow \mathsf{IaN}_2$,* etc
- *Class IVN is sent to itself.*
- *IS $\leftrightarrow$ IaS.*
- *IIS $\leftrightarrow$ IIaS.*

Proof. Choose one representant of each class, and count the numbers of intersection points and common tangents, real and imaginary, of each multiplicity. The result is displayed in table 2 below. Remark that no two classes present the same data, except for pairs of associated non-symmetric classes, and Classes IIIN and IIIS. The tangential map sends a class to some class where the data concerning intersections points have been exchanged with those concerning common tangents.

Remark also that if [the real locus of] some real proper conic C_1 lies inside another real proper conic C_2, then $C_2^\vee$ lies inside $C_1^\vee$. This is obvious after section 3. This shows that Class IIIN is mapped to itself, as is Class IIIS; this also determines the images of the pairs of non-symmetric classes: the two members are exchanged. □

6. An example of application

Let C_1 and C_2 be two real proper conics in configuration IS. Then the intersection of their insides has two connected components. One wants to find one point in each component, call them A and B.

We use the previous study to affirm that $C_1^\vee$ and $C_2^\vee$ are in configuration IaS, that is: two conics with no real intersection. The pencil they generate has three degenerate elements: one couple of real lines and two couples of conjugate imaginary lines (see [3]). The couple of real lines separates $C_1^\vee$ and $C_2^\vee$. Back in the primal space, it corresponds to a couple of real points A and B, one per component of the intersection of the insides of C_1 and C_2, as looked for.

The computations of A and B are performed as follows: let f and g be quadratic forms defining C_1 and C_2. Their tangential forms $\widetilde{f}$ and $\widetilde{g}$ have matrices $\widetilde{F}$ and $\widetilde{G}$. Now consider the characteristic polynomial of $\widetilde{F} + u\widetilde{G}$:

$$\mathrm{Disc}(y\,I - (\widetilde{F} + u\,\widetilde{G})) = y^3 - \nu_t(u)\,y^2 + \mu_t(u)\,y - \phi_t(u).$$

The unique parameter $u = u_0$ such that $\widetilde{F} + u\widetilde{G}$ defines a couple of real lines is characterized by:

$$\phi_t(u) = 0 \quad \wedge \quad \mu_t(u) < 0.$$

The polynomial ϕ_t has degree 3 in u, the polynomial μ_t has degree at most 2. This way we got an exact semi-algebraic description (see [1]) of the points A and B we have been looking for. It is of interest to note that our method works even if the equations of the conics depend of parameters.

To make the method clearer, we consider a concrete example. Consider the affine conics defined by the equations in x, y

$$\begin{gathered}(x^2 + y^2 - 4) + t\,xy + t^2\,x^2 = 0,\\ (xy - 1) + t\,(x^2 - y^2) + t^2 = 0,\end{gathered}$$

depending on the parameter t. They are obviously deformations of a circle and an hyperbola in configuration IS. We homogenize the equations to fit in our projective setting; thus we consider

$$\begin{aligned} f = & \quad (x^2 + y^2 - 4z^2) + t\,xy + t^2\,x^2, \\ g = & \quad (xy - z^2) + t\,(x^2 - y^2) + t^2\,z^2. \end{aligned}$$

One finds, proceeding as explained in [3], that the two conics are in configuration IS exactly when $P(t) < 0$, where

$$P(t) = 3\,t^6 - 16\,t^5 - 2\,t^4 + 8\,t^3 - 69\,t^2 + 8\,t - 12.$$

The polynomial P has two real roots, one between 5 and 6, the other one between -2 and -1.

One finds for $A(t)$ and $B(t)$ the points $[x(t) : y(t) : z(t)]$ where the $(x(t)X + y(t)Y + z(t)Z)$ are the factors of $\widetilde{f} + u(t)\,\widetilde{g}$, and $u(t)$ is the analytic function solution of $\phi(u) = 0$ such that $\mu(u) < 0$, where

$$\begin{aligned} \phi(u) \quad &= A_3(t)\,u^3 + A_2(t)\,u^2 + A_1(t)\,u + A_0(t), \\ \mu(u) \quad &= B_2(t)\,u^2 + B_1(t)\,u + B_0(t), \end{aligned}$$

with

$$\begin{aligned} A_3(t) \quad &= -t^8 + 3/2\,t^6 - 1/16\,t^4 - 3/8\,t^2 - 1/16, \\ A_2(t) \quad &= 3/4\,t^8 + 4\,t^7 - 5/16\,t^6 - t^5 \\ &\quad -11/8\,t^4 - 5/2\,t^3 + 11/16\,t^2 - 1/2\,t + 1/4, \\ A_1(t) \quad &= -3\,t^7 - 5/2\,t^5 + 12\,t^4 + 7/2\,t^3 + 19\,t^2 + 2\,t + 4, \\ A_0(t) \quad &= 9\,t^4 + 24\,t^2 - 16. \end{aligned}$$

and

$$\begin{aligned} B_2(t) \quad &= -t^6 + 7/4\,t^4 - 1/2\,t^2 - 1/4, \\ B_1(t) \quad &= 4\,t^5 + 4\,t^4 - 2\,t^3 + 2 + 9\,t^2 - 2\,t, \\ B_0(t) \quad &= -3\,t^4 + 2\,t^2 + 8. \end{aligned}$$

Suppose one wants to find the first terms of the Puiseux expansion of this solution for t near from 3. Set $s = t - 3$. For $s = 0$, one finds that the u which makes $\phi = 0$ and $\mu < 0$ is

$$u_0 = \frac{57 - 2\,\sqrt{1099}}{74}.$$

After examination of the Newton diagram of $F(v, s) = \phi(u_0 + v, 3 + s)$, one sees that the Puiseux expansion of $u(s)$ is actually a Taylor series. One calculates easily the first terms:

$$\begin{aligned} u(s) = u_0 + \left(\frac{5725}{81326} - \frac{85413}{162652}\,u_0 \right)\, s \\ + \left(\frac{7584189175}{26455673104}\,u_0 - \frac{854202513}{13227836552} \right)\, s^2 + O\left(s^3\right) \end{aligned}$$

One deduces that the two affine points $(x(s), y(s))$ are such that

$$\begin{aligned} -x(s)^2 \quad &= -C_1(s)\,/\,C_4(s), \\ -y(s)^2 \quad &= -C_3(s)\,/\,C_4(s), \\ -2x(s)\,y(s) \quad &= \quad C_2(s)\,/\,C_4(s). \end{aligned}$$

with

$$\begin{array}{ll} C_1(s) & = -4 + (-24 - s^3 - 9\,s^2 - 26\,s)\,u, \\ C_2(s) & = (-8 - s^2 - 6\,s)\,u + 12 + 4\,s, \\ C_3(s) & = (24 + s^3 + 9\,s^2 + 26\,s)\,u - 40 - 4\,s^2 - 24\,s, \\ C_4(s) & = (-37/44 - s^2 - 6\,s)\,u + 31/44 + 3/4\,s^2 + 9/2\,s. \end{array}$$

One deduces that

$$\begin{array}{ll} x(s) & = \varepsilon\sqrt{-C_1(s)\,/\,C_4(s)}, \\ y(s) & = -\varepsilon\sqrt{-C_3(s)\,/\,C_4(s)}, \end{array}$$

with $\epsilon = \pm 1$. It is easy to get the first terms of the Taylor expansions. For instance,

$$\begin{array}{ll} x(s) & = x_0\ \left(1 + x_1\ s + x_2\ s^2 + O\left(s^3\right)\right), \\ y(s) & = y_0\ \left(1 + y_1\ s + y_2\ s^2 + O\left(s^3\right)\right), \end{array}$$

with

$$x_0 = 4\sqrt{-\frac{892\,u_0 + 82}{4371}}, \qquad x_1 = -\frac{34136731}{192149160} + \frac{8324536}{24018645}\,u_0,$$
$$x_2 = \frac{56467556991933}{1897344567336260}\,u_0 + \frac{1409352180577583}{60715026154760320},$$

and

$$y_0 = -4\sqrt{\frac{-736\,u_0 + 1226}{4371}}, \qquad y_1 = \frac{678535}{38429832} - \frac{2848157}{9607458}\,u_0,$$
$$y_2 = -\frac{630387301227139}{36429015692856192} + \frac{642658831396033}{4553626961607024}\,u_0.$$

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