Diagonally symmetric polynomials of the roots of some systems of equations

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March 31^{st}, 2006

Abstract

We present two situations in which the diagonally symmetric polynomials of the n roots of a system of polynomial equations depend in a simple way on the coefficients of the equations: for Gröbner bases with prescribed leading terms, and for zero–dimensional strict complete intersections. The results are independent on the characteristic of the ground field.

Introduction

The symmetric polynomials of the roots of a monic univariate polynomial are the elements of the ring generated by its coefficients (that are, up to the sign, the elementary symmetric polynomials in the roots). At the end of the XIX-th century, illustrious mathematicians (Schläfli [6], Cayley [4]) looked for similar results concerning systems of polynomial equations with finitely many solutions. They introduced, as appropriate analogs of the symmetric polynomials, the *diagonally symmetric polynomials.* In this note we present some situations in which the diagonally symmetric polynomials in the roots of some systems of polynomial equations can be expressed simply in terms of the coefficients of the equations.

1 Diagonally Symmetric Polynomials

The *diagonally symmetric polynomials* are the polynomial invariants of $r \times n$ matrices under the symmetric group $\mathfrak{S}_n$ acting by permutation of the columns of the matrices. Let $x_i(a_j)$ be the coefficients of the matrix, where the i's are the row indices and the j's are the column indices. We can see the a_j's as points in the r–dimensional affine space, and the x_i's as the coordinate functions. The diagonally symmetric polynomials then appear as the polynomial functions on multisets of points of length n. Some classical families of symmetric polynomials have diagonally symmetric analogues.

The *elementary diagonally symmetric polynomials* e_α for $\alpha \in \mathbb{N}^{r*}$ (the set of non–zero vectors with r non–negative integer coordinates) and $|\alpha| \leq n$ are defined by the generating function:

$$1 + \sum_{\alpha} e_\alpha \mathbf{t}^\alpha = \prod_{j=1}^{n} L(a_j)$$

where L is the linear form $1 + t_1x_1 + \ldots + t_rx_r$. This generating function is known in computer algebra as the (deshomogenized) *Chow form* oof the multiset of the a_j's.

Th *power sums* p_α for $\alpha \in \mathbb{N}^{r*}$ are $p_\alpha = \sum_{j=1}^{n} \mathbf{x}^\alpha(a_j)$. If the a_j's are the solutions of a system of equations, these objects are known as the *traces of the monomials*.

The *monomial functions* are the sums of all monomials in the $x_i(a_j)$'s in some orbit under $\mathfrak{S}_n$. They can be defined as well by means of a generating function in indeterminates u_α, $\alpha \in \mathbb{N}^{r*}$:

$$\sum_{\vec{\lambda}} m_{[\vec{\lambda}]} u^{[\vec{\lambda}]} = \prod_{j=1}^{n} S(a_j) \qquad (1)$$

where S is the polynomial $1 + \sum_{\alpha \in \mathbb{N}^{r*}} u_\alpha \mathbf{x}^\alpha$; the $\vec{\lambda}$'s are the multisets of elements of $\mathbb{N}^{r*}$ ("vector partitions" or "multipartite partitions") of length at most n; $u^{[\vec{\lambda}]}$ is the product of the u_α's for α term of $\vec{\lambda}$. The combinatorialist will identify it as an avatar of (a diagonally symmetric analogue of) a *Cauchy formula*. The computer algebraist will recognize in its truncations (obtained from $S_\Omega = 1 + \sum_{\alpha \in \Omega} u_\alpha \mathbf{x}^\alpha$, Ω finite) the *generalized Chow forms* of the a_j's.

2 Diagonally symmetric functions of the roots of systems of equations

We consider systems of polynomial equations in the indeterminates $x_1, \ldots, x_r$ with coefficients in some field $\mathbb{K}$. The solutions of the system are always considered in $\mathbb{L}^n$, where $\mathbb{L}$ is an algebraically closed extension of $\mathbb{K}$. Suppose we fix some special shape to the systems, so that they all have finitely many zeros, the same number n (taking into account multiplicities). It is then meaningful to consider the diagonally symmetric polynomials of the roots and to inquire about the way they depend on the coefficients of the system.

Note that if we want characteristic–free answers, we should be careful. For instance, it is not enough to find formulas expressing the power sums in terms of the coefficients. Indeed, as in the case of symmetric polynomials, the power sums don't generate the ring of diagonally symmetric polynomials in small characteristic. But worst, even the elementary diagonally symmetric polynomials may not be a generating family ! (see [3]). On the contrary, the monomial functions are always a linear basis, thus it is enough to know how to express the monomial function in terms of the coefficients.

We present from [2] results for two kinds of systems: Gröbner bases with prescribed leading terms, and zero-dimensional strict complete intersection. In both cases, the strategy consists in using the fact that a Generalized Chow Form is the determinant of an endomorphism of the affine coordinate ring of the zeros of the system: the endomorphism of multiplication by the corresponding S_Ω. This permits us to provide a formula for each monomial function in terms of the coefficients of the system.

2.1 Gröbner bases

Let $\preceq$ be a monomial order on the monomials in $x_1, \ldots, x_r$. Let $\mathbf{x}^{\beta(1)}, \ldots, \mathbf{x}^{\beta(k)}$ be a collection of monomials, containing a power of each indeterminate. We consider the systems of equations:

$$f_1 = \ldots = f_k = 0$$

where

$$f_i = \mathbf{x}^{\beta(i)} + \sum_{\alpha : \mathbf{x}^\alpha \prec \mathbf{x}^{\beta(i)}} a_\alpha^{(i)} \mathbf{x}^\alpha.$$

These systems are the points of some affine space A (the coordinates are the $a_\alpha^{(i)}$'s). Consider those systems that are a Gröbner basis (for $\preceq$) of the ideal they generate. They form an algebraic subset G of A (see [2], Corollaire 4.4). Moreover, all systems in G have the same number of solutions, say n (the cardinal of the set M of monomials that can't be divided by any of the $\mathbf{x}^{\beta(i)}$'s). Then it is meaningful to consider the diagonally symmetric polynomials of the roots of the systems in G. Then [2]:

Theorem 1. *Any diagonally symmetric polynomial in the roots of the systems in G is a polynomial function on G.*

2.2 Strict complete intersections

A strict complete intersection is a zero–dimensional ideal that is a complete intersection and have no zero at infinity in some compactification of the affine space (a *weighted projective space*). The existence of an associated resultant provides us explicit formulas for the diagonally symmetric polynomials of the roots.

More precisely, let $\omega \in (\mathbb{N}^*)^r$. It defines a graduation on $\mathbb{K}[x_1, \ldots, x_r]$ determined by: $\deg_\omega X_i = \omega_i$ for all i. The ideal I is a strict complete intersection for the weight ω in it admits a generating set $\{f_1, \ldots, f_r\}$ such that the zero locus of $h_1, \ldots, h_r$, the leading homogeneous components of the f_i's, have $\mathbf{0}$ as only common zero.

Let $\mathbf{d} = (d_1, \ldots, d_r) \in (\mathbb{N}^*)^r$. Let B be the set of all polynomial systems $f_1, \ldots, f_r$ as above with h_i having (weighted) degree d_i. Let Δ be the anisotropic resultant [5] associated to the weight ω and degrees $\mathbf{d}$, evaluated the h_i's. Let W be the Zariski open subset of B defined by $\Delta \neq 0$. This is exactly the set of the systems in B that are a presentation of a strict complete intersection for ω. Then all the systems in W have the same number of solutions (couting multiplicities), that is $n = \prod_i (d_i/\omega_i)$; we can consider the diagonally symmetric polynomials of the roots of these systems. Then we have ([2]), as a consequence of the Poisson formula for anisotropic resultants:

Theorem 2. *Any diagonally symmetric polynomial in the roots of the systems in W is of the form P/Δ^s for some polynomial function P of the coefficients and non–negative integer s.*

2.3 Systems with both properties

Interestingly, for zero–dimensional systems that are at the same time a monic Gröbner basis of the ideal they generate, and a presentation of this ideal as a strict complete intersection, much more is known about the diagonally symmetric polynomials of the roots (in characteristic zero). Such systems are those of the form $f_1 = \ldots = f_r = 0$ where each f_i has as leading term (for some fixed monomial order) some power of x_i. Aizenberg and Kytmanov [1] showed that for these systems, the power sums of the roots are obtained as the coefficients of some series expansion of the rational function $x_1 \cdots x_r\, J/f_1 \cdots f_r$, where J is the Jacobian determinant of the f_i's. Consider specially the *triangular systems*

$$g_1(x_1) = g_2(x_1, x_2) = \ldots = g_r(x_1, \ldots, x_r) = 0$$

where each g_i, seen as polynomial in x_i, has leading coefficient $\ell_i(x_1, \ldots, x_{i-1})$ such that the systems $g_1 = \ldots = g_{i-1} = \ell_i = 0$ have no solution. Using Aizenberg-Kytmanov's identity, one gets that for triangular systems with prescribed degrees, the power sums are rational functions of the coefficients.

Let us consider an example: $r = 2$, with $x = x_1$, $y = x_2$, and $g_1 = a_2x^2 + a_1x + a_0$, $g_2 = (q_2x^2 + q_1x + q_0)y^2 + (s_2x^2 + s_1x + s_0)y + (t_2x^2 + t_1x + t_0)$. Then the power sum p_{11} is $N/(a_2R)$, where

$$\begin{aligned} N = {} & a_1s_0{a_2}^2q_0 + a_1s_0a_2q_2a_0 - 3\,a_0a_1s_2a_2q_0 \\ & + s_2{a_0}^2q_2a_1 + a_0s_1a_2a_1q_1 + 2\,a_0s_1{a_2}^2q_0 - 2\,s_1a_2q_2{a_0}^2 - 2\,a_0{a_2}^2s_0q_1 \\ & + 2\,a_2s_2{a_0}^2q_1 + {a_1}^3s_2q_0 - a_0{a_1}^2s_2q_1 - {a_1}^2s_1a_2q_0 \end{aligned}$$

and R is the resultant of g_1 and $\ell_2 = q_2x^2 + q_1x + q_0$.

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