Reduced Kronecker coefficients

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Abstract

We provide a formula that recovers the Kronecker coefficients (the multiplicities of the irreducible representations in the tensor products of two irreducible representations of the symmetric group) from the reduced Kronecker coefficients (limits of certain stationary sequences of Kronecker coefficients introduced by Murnaghan). This formula generalizes a formula due to Rosas for Kronecker coefficients indexed by two two–row shapes.

We use our formula to obtain a new stability bound for the Kronecker coefficients, and to describe explicitely the Kronecker coefficients indexed by two two–row shapes as a piecewise quasi–polynomial, with the chambers of a fan as domains of quasi–polynomiality.

1 The rational convex polyhedra approach to Representation Theory

A fundamental problem in Group Representation Theory is the following: given a linearly reductive group G, describe the coefficients $m_{\mu\nu}^{\lambda}$ in the decomposition into irreducibles of a product of two (finite–dimensional, complex) irreducible representations:

$$V_\mu \otimes V_\nu = \bigoplus_{\lambda} m_{\mu\nu}^{\lambda} V_\lambda$$

While this problem is solved satisfyingly in several ways for the general linear group $G = GL_n(\mathbb{C})$ (the most elementary Lie group), this is not the case for the symmetric group $G = \mathfrak{S}_n$ (the most fundamental finite group).

For $G = GL_n(\mathbb{C})$, the multiplicities $m_{\mu,\nu}^{\lambda} = c_{\mu,\nu}^{\lambda}$ are called the *Littlewood–Richardson coefficients*. They are described in a purely combinatorial way by the *Littlewood–Richardson rule*, established in 1934 (although not proved before 1977), see for instance [9]. In 1992, another type of combinatorial description was proposed by Berenstein and Zelevinsky [1]: they showed that the Littlewood–Richardson coefficients count the integral points in polytopes of a well–defined family. This started a series of fruitful works by several authors describing features of the Littlewood–Richardson coefficients in the language of combinatorics of rational convex polyhedra. Let LR_n be the set of triples $(\lambda, \mu, \nu) \subset \mathbb{Z}^{3n}$ such that $c_{\mu,\nu}^{\lambda} > 0$. It was proved that LR_n is a saturated finitely generated semigroup, and an explicit description of the facets of the polyhedral convex cone $\mathsf{LR}_n^{\mathbb{R}}$ it generates in $\mathbb{R}^{3n}$ was obtained. This was done by relating the problem of describing $\mathsf{LR}_n^{\mathbb{R}}$ with Horn's Conjecture about sums the eigenvalues of the sum of two hermitian matrices [7] and then proving the celebrated Saturation Conjecture [8]. Finally, it was shown that the Littlewood–Richardson coefficients $c_{\mu\nu}^{\lambda}$ depend piecewise quasi–polynomially[1] on the parts of λ, μ, ν, with the chambers (maximal cones) of a fan subdividing $\mathsf{LR}_n^{\mathbb{R}}$ as pieces [12].

Strickingly, no combinatorial description of the coefficients $m_{\mu\nu}^{\lambda} = g_{\mu\nu}^{\lambda}$ (*Kronecker coefficients*) associated to the symmetric group $G = \mathfrak{S}_n$ is known[2]. The long string of successes in working out the description of the Littlewood–Richardson coefficients in the language of rational convex polyhedra has suggested to attempt a similar approach for the Kronecker coefficients. Let $\mathsf{Kron}_{\ell_1,\ell_2}$ be the set of triples (λ, μ, ν) of partitions with $\ell(\mu) \leq \ell_1$, $\ell(\nu) \leq \ell_2$. In 2007, Christandl, Harrow and Mitchison [4] showed that $\mathsf{Kron}_{\ell_1,\ell_2}$ is a finitely generated semigroup. It is, however, not saturated. Kirillov [5] and Klyachko [6] suggested that the saturation property may hold for a related family, the *reduced Kronecker coefficients* $\overline{g}_{\alpha,\beta}^{\gamma}$, that are limits of certain stationary sequences of Kronecker coefficients.

We present a formula that recovers the Kronecker coefficients from the reduced Kronecker coefficients. Using this formula, we improve a bound given previously by Ernesto Vallejo [17] for the stabilisation of the stationary sequences of Kronecker coefficients just mentioned. We also describe a fan and a quasi–polynomial formula on each of its chambers for the reduced Kronecker coefficients $\overline{g}_{(r)(s)}^{\gamma}$. From these results we deduce deduce quasi–polynomial formulas on the chambers of a fan for the Kronecker coefficients $\overline{g}_{\mu\nu}^{\lambda}$ where μ and ν have length at most 2.

2 Reduced Kronecker coefficients

The irreducible representations V_λ of the symmetric group $\mathfrak{S}_n$ are indexed with the partitions λ with sum $|\lambda| = n$. The Kronecker coefficients $g_{\mu\nu}^{\lambda}$ are thus indexed by triples of partitions (λ, μ, ν) with $|\lambda| = |\mu| = |\nu|$. Given a partition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, denote with $\alpha[n]$ the sequence $(n - |\alpha|, \alpha_1, \alpha_2, \ldots, \alpha_k)$. Murnaghan [11] showed that given any three partitions α, β and γ, the sequence of Kronecker coefficients $g_{\alpha[n],\beta[n]}^{\gamma[n]}$ is stationary. Its limit (stable value) is called the *reduced Kronecker coefficient* $\overline{g}_{\alpha\beta}^{\gamma}$. The reduced Kronecker coefficients are actually the structural constants for a linear basis for the polynomials in countably many variables (the *character polynomials*, see [9]). Interestingly, they also generalize the Littlewood–Richardson coefficients: $\overline{g}_{\alpha\beta}^{\gamma} = c_{\alpha\beta}^{\gamma}$ when $|\alpha| + |\beta| = |\gamma|$.

It follows from the semigroup property for the Kronecker coefficients [4], that for any fixed ℓ_1 and ℓ_2, the triples of partitions (α, β, γ) such that $\overline{g}_{\alpha\beta}^{\gamma} > 0$, $\ell(\alpha) \leq \ell_1$, and $\ell(\beta) \leq \ell_2$ is a finitely generated semigroup. That it is saturated is an open conjecture formulated by

[1] A quasi–polynomial is a function f on $\mathbb{Z}^n$ such that, for some full rank sublattice $\mathcal{L}$ and for some polynomials P_c attached to each coset of $\mathcal{L}$, there is $f(x) = P_c(x)$ when $x \in c$.

[2] Stanley, in [16], p. 539 : "One of the main open problems in the combinatorial representation theory of $\mathfrak{S}_N$ is to obtain a combinatorial interpretation of $g_{\mu\nu}^{\lambda}$ in general."

Kirillov [5] and Klyachko [6].

We established the following formula that recovers the Kronecker polynomials from the reduced Kronecker polynomials.

Theorem 1 ([3]). *Let λ, μ, ν be partitions of the same integer. Then:*

$$g_{\mu\nu}^{\lambda} = \sum_{i=1}^{\ell(\mu)\ell(\nu)} (-1)^{i+1} \overline{g}_{\mu^{\dagger 1}\,\nu^{\dagger 1}}^{\lambda^{\dagger i}} \tag{1}$$

where $\lambda^{\dagger i}$ is the partition obtained from λ by incrementing the $i-1$ first parts and removing the i–th part.

We deduced from it a new bound for the stability of the sequences $(g_{\alpha[n],\beta[n]}^{\gamma[n]})_n$. This improved a previous result by Ernesto Vallejo [17].

Theorem 2 ([2]). *Let α, β, γ be three partitions. Then $g_{\alpha[n],\beta[n]}^{\gamma[n]} = \overline{g}_{\alpha\beta}^{\gamma}$ for all n greater than or equal to*

$$\max(|\alpha| + |\beta| + \gamma_1, |\alpha| + \alpha_1, |\beta| + \beta_1, |\gamma| + \gamma_1)$$

3 The case of two two–row shapes

Formulas for the Kronecker coefficients are known in few particular cases. Remmel and Whitehead [13] derived formulas for the coefficients $g_{\mu\nu}^{\lambda}$ when μ and ν have at most two parts ("two–row shapes"). Later, the third author used Sergeev's formula to describe the same Kronecker coefficients as the difference of number of integral points in two convex rational polygons Γ^+ and Γ^- defined by inequalities $aX + bY + c \geq 0$ where the constant terms c depend affinely on the parts of λ, μ and ν [14].

Considering reduced Kronecker coefficients (as advocated in [15]) simplifies the presentation of these results. Formula (1) applies as follows for Kronecker coefficients indexed by two two–row shapes:

$$g_{(N-r,r)(N-s,s)}^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)} = \overline{g}_{(r)(s)}^{\lambda_2,\lambda_3,\lambda_4} - \overline{g}_{(r)(s)}^{\lambda_1+1,\lambda_3,\lambda_4} + \overline{g}_{(r)(s)}^{\lambda_1+1,\lambda_2+1,\lambda_4} \tag{2}$$

The first term $\overline{g}_{(r)(s)}^{\lambda_2,\lambda_3,\lambda_4}$ counts the integral points in Γ^+, and the other two count the integral points in two polygons Γ_2 and Γ_3 such that $\Gamma_3 \subset \Gamma_2$ and $\Gamma_2 \setminus \Gamma_3 = \Gamma^-$. Thus Formula (1) is a generalization of the formula in [14].

Using this interpretation of the reduced Kronecker coefficients $\overline{g}_{(r)(s)}^{\gamma}$, we described in explicitely in [3] a fan $\mathcal{F}$ of $\mathbb{R}^5$, and for each of its chambers σ, a quasi–polynomial $q_\sigma(r, s, \gamma_1, \gamma_2, \gamma_3)$ giving $\overline{g}_{(r)(s)}^{\gamma}$ on σ. From this result and (2), and by taking profit of the fact that the quasi–polynomials q_σ coincide not only on the borders of their chambers but on larger areas, we were able to compute explicitely a fan $\mathcal{F}'$ and quasi–polynomials formulas $p_\sigma(n, r, s, \lambda_2, \lambda_3, \lambda_4)$ giving the Kronecker coefficients $g_{(n-r,r)(n-s,s)}^{\lambda}$ on each chamber σ of $\mathcal{F}'$. This existence of such formulas was proved by Mulmuley [10] but our work is their first explicit computation in a non–trivial example. These formulas allowed us to check, in the particular case of Kronecker coefficients indexed by two two–rows shapes, a series of conjectures by Mulmuley giving support to his Geometric Complexity Theory [10].

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