MILNE'S VOLUME FUNCTION AND VECTOR SYMMETRIC POLYNOMIALS

EMMANUEL BRIAND AND MERCEDES ROSAS

Abstract. The number of real roots of a system of polynomial equations fitting inside a given box can be counted using a vector symmetric polynomial introduced by P. Milne, the volume function. We provide the expansion of Milne's volume function in the basis of monomial vector symmetric functions, and observe that only monomial functions of a particular kind appear in the expansion, the squarefree monomial functions.

By means of an appropriate specialization of the vector symmetric Newton identities, we derive an inductive formula that expresses the squarefree monomial functions in the power sums basis. As a corollary, we obtain an inductive formula that writes Milne's volume function in the power sums basis. The lattice of the sub–hypergraphs of an hypergraph appears in a natural way in this setting.

1. Introduction

Given a system of polynomial equations in r variables with finitely many complex roots, consider the problem of counting its real roots lying in a given box of $\mathbb{R}^r$. In [Milne(1992)], Philip Milne introduced a function he called the *volume function* to deal with this problem. This function was used in [González–Vega and Trujillo(1997), Carreras and González–Vega(2004)] afterwards. Milne's volume function is a polynomial function of n (non necessarily distinct) points of $\mathbb{C}^r$ with coefficients in $\mathbb{Z}[u, x_1, \ldots, x_r]$, where u, x_1, $\ldots$, x_r are independent parameters. It is a *vector symmetric function*, meaning that it doesn't depend on the order of the n points. The natural question of decomposing Milne's volume function in the classical bases of vector symmetric functions has not been tackled. In this note, Milne's volume function is decomposed in the basis of the *monomial vector symmetric functions*. All monomial vector symmetric functions having non–zero coefficient in the decomposition happen to be of a special kind: *squarefree monomial functions*. The note provides an inductive formula to compute the squarefree monomial functions from another basis of the vector symmetric functions, the *power sums*.

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The note is organized as follows. Section 2 reviews the definition of Milne's volume function. Section 3 introduces the vector symmetric functions. Section 4 provides the expansion of Milne's volume function in the basis of the monomial functions. Finally, Section 5 presents and explains the inductive formula that can be used to decompose the squarefree monomial functions in power sums.

2. Milne's volume function

Let r and n be positive integers. Consider an $r \times n$ matrix A whose entries $a_{i,j}$, $i = 1, \ldots, r$, $j = 1, \ldots, n$ are independent variables. Consider $r+1$ additional independent variables u, $x_1, \ldots, x_r$.

Definition 1. [Milne(1992)] Milne's volume function *is defined as*

$$V(u, x_1, x_2, \ldots, x_r, A) = \prod_{j=1}^{n} \left(u + \prod_{i=1}^{r} (x_i - a_{i,j}) \right). \tag{1}$$

We shortly explain the origin of the volume function. It was introduced in Milne's paper [Milne(1992)] to deal with the following problem. Consider (S), a system of polynomial equations in r variables, with real coefficients, and finitely many complex solutions. Let n be the total number of solutions, counted with their algebraic multiplicity. One wants to locate the real solutions of (S). We quote the introduction of [Milne(1992)]:

> "In one dimension it is common to implement this procedure in two phases: an isolation phase and an approximation phase. The isolation phase produces a set of intervals sufficiently small for there to exist a single solution in each of them and is often implemented by recursively dividing a bounding interval given some strategy for counting the number of solutions inside an arbitrary interval. The second phase takes each of the intervals and uses numerical techniques to approximate the solution to some given tolerance.
>
> It is possible to use this strategy in many dimensions as well, provided that an analogous technique for counting the number of solutions that lie within an n-dimensional rectangle or box is available."

In the univariate setting, Sturm sequences provide a strategy to count the real roots inside an interval. Milne provides a sequence of polynomials $P_i(x_1, x_2, \ldots, x_r)$ attached to (S), that can be used analogously to Sturm sequences, for counting the real roots of (S) in a box. The polynomials $P_i(x_1, x_2, \ldots, x_r)$ are the subresultants of the pair of polynomials in u: $(V, \partial V/du)$, where V is the volume function with A evaluated at the matrix $A(S)$ whose columns represent the roots of (S).

There exists another multivariate generalization of Sturm sequences, due to Paul Pedersen [Pedersen(1991b)]. It is further studied with co–authors in [Pedersen et al.(1993)]. Pedersen's approach is more general: it allows to count the real solutions of a system (S) in an arbitrary semi–algebraic set P, not necessarily a box. In case P is a box, Pedersen's and Milne's constructions are still slightly different, and are compared in section 3 of [Pedersen(1991b)].

3. Vector symmetric functions

Consider once again the $r \times n$ matrix A of indeterminates $a_{i,j}$, $i = 1, \ldots, r$, $j = 1, \ldots, n$. A polynomial in the rn variables $a_{i,j}$ is said to be *vector symmetric*[1] if it is unchanged under permutations of the columns of A. Vector symmetric polynomials with coefficients in the ring R form an algebra that we denote with $VSym_R(A)$. This algebra inherits a (multi)grading with values in $\mathbb{N}^r$ from the grading of the ambient algebra of polynomials defined by giving to $a_{i,j}$ the multidegree $(0, \ldots, 0, 1, 0, \ldots, 0)$ (a 1 is in i–th position, all other coefficients are 0).

It is easy to see that Milne's volume function is a vector symmetric function with coefficients in the ring $\mathbb{Z}[u, x_1, x_2, \ldots, x_r]$.

It seems that vector symmetric polynomials were first introduced by Schläfli in his work on resultants [Schläfli(1852)]. Systematic studies of these objects were later undertaken by MacMahon [MacMahon(1916)] and Junker [Junker(1893)]. Modern presentations can be found in [Dalbec(1999), Rosas(2001), Briand(2002), Rota and Stein(2005), Vaccarino(2005)].

We introduce some notations to define some remarkable families of vector symmetric polynomials. Let $\mathbb{N}^r$ be the set of vectors $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ with nonnegative integer coefficients. For such a vector, set $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_r$. The *elementary functions* e_α, for $\alpha \in \mathbb{N}^r$ such that $0 < |\alpha| \leq n$, are defined by their generating function:

$$1 + \sum_{\alpha} e_\alpha t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_r^{\alpha_r} = \prod_{j=1}^{n} \left(1 + a_{1,j} t_1 + a_{2,j} t_2 + \cdots + a_{r,j} t_r\right). \tag{2}$$

The *power sums* p_α, for $\alpha \in \mathbb{N}^r$ such that $|\alpha| > 0$, are defined by:

$$p_\alpha = \sum_{j=1}^{n} a_{1,j}^{\alpha_1} a_{2,j}^{\alpha_2} \cdots a_{r,j}^{\alpha_r}.$$

Both families generate $VSym_R(A)$ as a R–algebra if R contains the rational numbers. They are related by the following analogues of the Newton Formulas: for all $\gamma \in \mathbb{N}^r$ with $|\gamma| > 0$,

$$|\gamma| \cdot e_\gamma + \sum_{\alpha+\beta=\gamma, \alpha\neq(0,0,\ldots,0)} (-1)^{|\alpha|} \binom{|\alpha|}{\alpha} p_\alpha e_\beta = 0. \tag{3}$$

[1] Other names for the same objects are: *multisymmetric polynomials*, *diagonal invariants of the symmetric group*, *MacMahon symmetric functions.*

where α and β are in $\mathbb{N}^r$ and one sets $e_\beta = 0$ for $|\beta| > n$, $e_{(0,0,\dots,0)} = 1$, $\binom{|\alpha|}{\alpha}$ is the multinomial coefficient $|\alpha|!/\alpha_1!\alpha_2!\cdots\alpha_r!$. These formulas are obtained from the classical Newton Formulas by polarization, see [Dalbec(1999)] (proof of Theorem 1.3) or [Briand(2002)].

The simplest linear basis of $VSym_R(A)$ is provided by the *monomial functions*, that are simply the orbit sums of monomials under permutations of the columns of A. More precisely, associate to each monomial $\prod_{i,j} a_{i,j}^{\mu_{i,j}}$ its exponent matrix M (the $r \times n$ matrix of the exponents $\mu_{i,j}$) and write the monomial as A^M. For a $r \times n$ matrix M with nonnegative integer entries whose columns are weakly decreasing with respect to the lexicographic order, define the monomial function $m[M]$ as:

$$m[M] = \sum A^{M'}$$

where M' runs into the orbit of M under permutations of columns. The multidegree of $m[M]$ is the vector of the row sums of M.

For instance, when $r = n = 2$, we have:

$$m\begin{bmatrix}2 & 1\\ 3 & 0\end{bmatrix} = A^{\begin{bmatrix}2 & 1\\ 3 & 0\end{bmatrix}} + A^{\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}} = a_{11}^2 a_{12} a_{21}^3 + a_{11} a_{21}^2 a_{22}^3$$

while

$$m\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} = A^{\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}} = a_{11} a_{12} a_{21} a_{22}.$$

When dealing with zero–dimensional systems of equations, one may consider the vector symmetric polynomials of their roots. Two such objects appear recurrently in the literature about zero–dimensional system solving: *Van der Waerden's u-resultant* [van der Waerden(1931)] and the *family of the power sums of the roots*. Consider such a system of polynomial equations,

$$(S):\ f_1(x_1,\dots,x_r) = f_2(x_1,\dots,x_r) = \cdots = f_k(x_1,\dots,x_r) = 0$$

with exactly n complex roots (counted with multiplicities). If we display these solutions as the columns of an $r \times n$ matrix $A(S)$, then the vector symmetric polynomials of the roots are the evaluations of the vector symmetric polynomials at $A = A(S)$. Then, Van der Waerden's u–resultant is precisely the evaluation of the generating function (2) of the elementary functions at $A = A(S)$. It is also the determinant of the operator of multiplication by $1 + x_1t_1 + x_2t_2 + \cdots + x_rt_r$ in the finite–dimensional vector space $\mathcal{V}(S) = \mathbb{C}[x_1, x_2, \dots, x_r]/\langle f_1, f_2, \dots, f_k\rangle$. The power sums of the roots are the evaluations of the power sums p_α at $A = A(S)$. They are the traces of the operator of multiplication by the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_r^{\alpha_r}$ of $\mathcal{V}(S)$. See Chapters 2 and 3 of the book of Cox, Little, O'Shea [Cox et al.(1998)] for more about the computation and use of the multiplication operators in polynomial system solving. See also [Pedersen(1991a)] and Chapter 4 of [Briand(2002)] for several strategies for computing the vector symmetric functions of the roots of a zero–dimensional system of equations, involving Gröbner bases, multivariate residues or resultants.

4. The expansion of the volume function in monomial functions

In Paragraph 2.1 of [Pedersen(1991b)], Pedersen explains how to express the formulas involved in his multivariate Sturm theory in terms of the vector symmetric power sums of the roots of the considered system. In Milne's paper [Milne(1992)], some hints are given to compute the volume function, but no reduction to vector symmetric functions computations is indicated. We looked for such a reduction and found that the volume function has a nice, simple expression in the basis of the monomial functions. This expansion will be presented in this section.

To compute the volume function, we recall its definition:

$$V(u, x_1, x_2, \ldots, x_r, A) = \prod_{j=1}^{n} \left(u + \prod_{i=1}^{r} (x_i - a_{i,j}) \right). \tag{1}$$

We start with the remark that it is enough to compute

$$V(u, 1, 1, \ldots, 1, -A) = \prod_{i=j}^{n} \left(u + \prod_{i=1}^{r} (1 + a_{i,j}) \right), \tag{4}$$

because one can re–obtain the general volume function by the following homogenization formula:

$$V(u, x_1, x_2, \cdots, x_r, A) = (x_1 x_2 \cdots x_r)^n V(u', 1, 1, \ldots, 1, -A') \tag{5}$$

with $u' = u/(x_1 x_2 \cdots x_r)$ and A' is the matrix with entries $a'_{i,j} = -a_{i,j}/x_i$. In (4), for each i between 1 and r, the term $\prod_{i=1}^{r}(1 + a_{i,j})$ expands as

$$\sum_{\alpha \in \{0,1\}^r} \prod_{i=1}^{r} a_{i,j}^{\alpha_i}.$$

This allows to write the dehomogenized volume function as:

$$V(u, 1, 1, \ldots, 1, -A) = \prod_{j=1}^{n} \left(1 + u + \sum_{\substack{\alpha \in \{0,1\}^r \\ \alpha \neq (0,\ldots,0)}} \prod_{i=1}^{r} a_{i,j}^{\alpha_i} \right),$$

This simplifies into

$$V(u, 1, 1, \ldots, 1, -A) = \sum_{\ell=0}^{n} (1+u)^{n-\ell} \Phi(\ell),$$

where $\Phi(\ell)$ is the sum of all monomials A^M that are squarefree (equivalently: all the entries of M are either 0 or 1) and such that M has exactly ℓ non–zero columns.

For $\alpha \in \mathbb{N}^r$ and ℓ nonnegative integer, let $\Phi_\alpha(\ell)$ be the sum of all squarefree monomials A^M with multidegree α such that M has exactly ℓ non–zero columns. The polynomial $\Phi_\alpha(\ell)$ is vector symmetric. We say that a monomial function is *squarefree* if it is the orbit sum of a squarefree monomial. Otherwise stated, the monomial function $m[M]$ is squarefree if and only if all entries of M are either 0 or 1. Then, $\Phi_\alpha(\ell)$ is the sum of all squarefree monomial functions $m[M]$ of multidegree α such that M has exactly ℓ non–zero columns.

Set now $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_r^{\alpha_r}$, and set $\mathbf{1} = (1, 1, \dots, 1)$, so that $\mathbf{x}^{\mathbf{1}} = x_1 x_2 \cdots x_r$. Then

$$V(u, 1, 1, \dots, 1, -A) = \sum_{\alpha \in \mathbb{N}^r} \sum_{\ell=0}^{n} (1+u)^{n-\ell} \Phi_\alpha(\ell)$$

and this rehomogenizes using (5) in the formula given by the following proposition.

Proposition 1. *The volume function expands as a vector symmetric polynomial with coefficients in* $\mathbb{Z}[u, x_1, x_2, \dots, x_r]$ *as follows:*

$$V(u, x_1, x_2, \dots, x_r, A) = \sum_{\alpha \in \mathbb{N}^r} (-1)^{|\alpha|} \sum_{\ell=0}^{n} \mathbf{x}^{\ell \mathbf{1} - \alpha} (\mathbf{x}^{\mathbf{1}} + u)^{n-\ell} \Phi_\alpha(\ell) \tag{6}$$

As an example, consider the simplest non–trivial case, $n = r = 2$. In this case, all the non–zero functions $\Phi_\alpha(\ell)$ are monomial functions. One gets

$$\begin{aligned} V(u, x_1, x_2, A) &= (u^2 + x_1 x_2)^2 \\ &\quad + (u + x_1 x_2) \left(+m \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} - x_1 m \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - x_2 m \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \\ &\quad + x_2^2\, m \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + x_1^2\, m \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + x_1 x_2\, m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &\quad - x_2\, m \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - x_1\, m \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + m \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

We finish this section with a few remarks concerning the functions $\Phi_\alpha(\ell)$. First, it is not difficult to check that $\Phi_\alpha(\ell) \neq 0$ if and only if $\ell \leq n$, $|\alpha| \geq \ell$ and $\alpha_i \leq \ell$ for all i. Next, we observe that, as in the example, when $r = 2$, each non–zero function $\Phi_{(\alpha_1, \alpha_2)}(\ell)$ is one monomial function, and not a sum of various monomial functions. Precisely, $\Phi_{(\alpha_1, \alpha_2)}(\ell) = m[M]$ where M is the matrix with $|\alpha| - \ell$ columns equal to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\ell - \alpha_2$ columns equal to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\ell - \alpha_1$ columns equal to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the remaining $n - \ell$ columns zero. When

$r > 2$ this is not anymore the general rule, *e.g* for $n = 2$, $r = 3$,

$$\Phi_{(1,1,1)}(2) = m \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} + m \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} + m \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Note nevertheless that with respect to system solving, the case $r = 2$ is not exotic: it corresponds to the study of the intersections of plane algebraic curves.

As a last remark, we point out that the elementary symmetric polynomials appear among the functions $\Phi_\alpha(\ell)$. Namely for any $\alpha \in \mathbb{N}^r$ such that $|\alpha| \le n$, we have that $e_\alpha = \Phi_\alpha(|\alpha|)$. This implies that there is no way to compute the volume function of a system of equations (S) without computing its u–resultant. Indeed, the polynomials $\mathbf{x}^{\ell\mathbf{1}-\alpha}(\mathbf{x}^{\mathbf{1}} + u)^{n-\ell}$ that appear in formula (6) as coefficients of $\Phi_\alpha(\ell)$ are linearly independent, and thus the coefficients of the u–resultant are among the coefficients of $V(u, x_1, x_2, \ldots, x_r, A(S))$, seen as a polynomial in u, x_1, x_2, $\ldots$, x_r. Thus, replacing in the formula the functions $\Phi_\alpha(\ell)$ with their expression in the elementary functions e_α will imply no additional difficulty of evaluation. The same remark holds for the expression of the functions $\Phi_\alpha(\ell)$ in the power sums p_α, that we will investigate in the next section. Indeed, the Newton Identities (3) allow to switch quickly between the elementary functions e_α and the power sums p_α.

5. Inductive formulas

In the section we consider the problem of computing the squarefree monomial functions and the functions $\Phi_\alpha(\ell)$ from the power sums.

We start with the squarefree monomial functions. There is a general explicit formula to express any vector symmetric monomial function in the power sums[2]. It is essentially the same formula that the one that expresses the classical monomial symmetric functions in the power sums, see [Doubilet(1972)]. To the general explicit formula corresponds an inductive formula that is better suited for computational purpose, see [Dalbec(1999)]. We don't use this here because when reducing a squarefree monomial function, it makes appear non–squarefree monomial functions. For instance, for $n = 3$, $r = 2$, it yields:

$$m \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = p_{(1,1)}\, m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - m \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} - m \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}. \tag{7}$$

We will now design an inductive formula specifically for squarefree monomial functions. Anticipating the statement of this new formula, we present how it performs the reduction of the same squarefree monomial function as in

[2]See [Junker(1893)], and see [Rosas(2001)] for a combinatorial interpretation involving the Möbius function of the lattice of set partitions.

(7):

$$3\,m\begin{bmatrix}1&1&0\\1&0&1\end{bmatrix} = 6\,p_{(2,2)} + p_{(1,1)}\,m\begin{bmatrix}1&0\\0&1\end{bmatrix} + p_{(1,0)}\,m\begin{bmatrix}1&0\\1&1\end{bmatrix} + p_{(0,1)}\,m\begin{bmatrix}1&1\\1&0\end{bmatrix} - 2\,p_{(1,1)}\,m\begin{bmatrix}1&0\\1&0\end{bmatrix} - 2\,p_{(1,2)}\,m\begin{bmatrix}1&0\\0&0\end{bmatrix} - 2\,p_{(2,1)}\,m\begin{bmatrix}0&0\\1&0\end{bmatrix} \tag{8}$$

To derive this new formula, we consider vector symmetric polynomials in the entries of another matrix of indeterminates, $B = (b_{\omega,j})$ where the column indices j are the integers between 1 and n, but the row indices ω are the elements of $\Omega = \{0,1\}^r \setminus \{(0,0,\dots,0)\}$. We observe that the specialization $b_{\omega,j} \mapsto \prod_{i=1}^r a_{i,j}^{\omega_i}$ provides a morphism of algebras Ψ from $VSym_R(B)$ to $VSym_R(A)$. It sends the elementary functions in B exactly to the squarefree monomial functions in A, and the power sums in B to power sums in A. To be more precise we need more notations. For $\theta \in \mathbb{N}^\Omega$, set $\delta(\theta) = \sum_{\omega\in\Omega}\theta_\omega\omega \in \mathbb{N}^r$ and $|\theta| = \sum_{\omega\in\Omega}\theta_\omega \in \mathbb{N}$. If $|\theta| \le n$, set also M_θ for the $r \times n$ matrix whose entries are all either 0 and 1, whose columns are in weakly decreasing lexicographic order, and such that for any $\omega \in \Omega$, the column ω^t (the transpose of ω) appears exactly θ_ω times. Then,

$$\Psi(e_\theta) = m[M_\theta] \qquad \text{for all } \theta \in \mathbb{N}^\Omega \text{ such that } 0 < |\theta| \le n,$$
$$\Psi(p_\theta) = p_{\delta(\theta)} \qquad \text{for all } \theta \in \mathbb{N}^\Omega.$$

As an example, formula (8) is the image by Ψ of the Newton identity (3) corresponding to $\gamma = (1,1,1)$:

$$3\,e_{(1,1,1)} = 6\,p_{(1,1,1)} + p_{(1,0,0)}e_{(0,1,1)} + p_{(0,1,0)}e_{(1,0,1)} + p_{(0,0,1)}e_{(1,1,0)} - 2\,p_{(0,1,1)}e_{(1,0,0)} - 2\,p_{(1,0,1)}e_{(0,1,0)} - 2\,p_{(1,1,0)}e_{(0,0,1)} \tag{9}$$

The application of Ψ to the Newton identities (3) yields the general formula stated in the following proposition.

Proposition 2. *For any $\theta \in \mathbb{N}^\Omega$ such that $|\theta| \le n$, we have:*

$$|\theta| \cdot m[M_\theta] = \sum_{\rho+\sigma=\theta,\rho\neq 0} (-1)^{|\rho|-1}\binom{|\rho|}{\rho} p_{\delta(\rho)} m[M_\sigma] \tag{10}$$

where ρ and σ are in $\mathbb{N}^\Omega$.

There is an interesting way to read the formula. Given a finite set T with r elements, we define an *hypergraph H on T* as a multiset of non–empty subsets of T. These subsets are called the *edges* of H. The *degree of a vertex $t \in T$* is the sum of the multiplicities of the edges in H that contain t. The *degree of the hypergraph H* is the sum of the multiplicities of its edges.

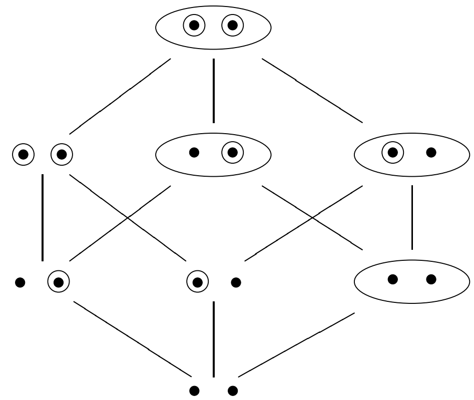

FIGURE 1. The lattice of the sub–hypergraphs of the hypergraph with edges $\{1,2\}$, $\{1\}$, $\{2\}$, all of multiplicity 1.

Note that our definition of hypergraph differs from the one in [Berge(1973)] because we allow multiple edges.

The set Ω can be identified with the set of the non–empty subsets of $\{1, 2, \ldots, r\}$. Any element $\theta \in \mathbb{N}^{\Omega}$ assigns to each non–empty subset of $\{1, 2, \ldots, r\}$ a multiplicity and thus defines an hypergraph H_θ on $\{1, 2, \ldots, r\}$. This hypergraph has degree $|\theta|$, and the vertex i has degree $\delta(\theta)_i$.

To each squarefree monomial function we can also associate an hypergraph, because such a monomial functions has necessarily the form $m[M_\theta]$ for some $\theta \in \mathbb{N}^{\Omega}$ such that $|\theta| \leq n$. The associated hypergraph is H_θ. Note that it admits M_θ as incidence matrix.

In formula (10), the monomial function at the left hand side represents the hypergraph H_θ, and in each term at the right–hand side corresponds to a proper sub–hypergraph H_σ. The multi–index $\delta(\rho)$ of the power sum in factor of $m[M_\sigma]$ measures the decreasing of degrees of the vertices when removing edges to H_θ to get H_σ (the degree of vertex i decreases by $\delta(\rho)_i$). The multinomial coefficient $\binom{|\rho|}{\rho}$ is the number of chains joining H_θ to H_σ in the poset of all sub–hypergraphs of H_θ. The poset of the sub–hypergraphs of the hypergraph corresponding to the monomial function $m\left[\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right]$ decomposed in Formula (8) is displayed in Figure 1.

We now derive from the induction formula for squarefree monomial functions an induction formula for the functions $\Phi_\gamma(\ell)$ that appear in Proposition 1. For $\alpha \in \mathbb{N}^r$ and $\ell \in \mathbb{N}$, set

$$c_\alpha(\ell) = \sum \binom{|\rho|}{\rho}$$

where the sum is over all $\rho \in \mathbb{N}^{\Omega}$ such that $|\rho| = \ell$ and $\delta(\rho) = \alpha$. This number $c_\alpha(\ell)$ is the total number of chains in the poset of hypergraphs on $\{1, 2, \ldots, r\}$, joining the minimal hypergraph (the hypergraph with no edge) to some hypergraph of degree ℓ fulfilling the condition: for each i, the vertex i has degree α_i.

Fix $\ell \in \mathbb{N}$ and $\gamma \in \mathbb{N}^r$. Summing (10) over all $\theta \in \mathbb{N}^\Omega$ such that $|\theta| = \ell$ and $\delta(\theta) = \gamma$ yields the formula stated in the following corollary.

Corollary 1. *Let $\ell \in \mathbb{N}$ and $\alpha \in \mathbb{N}^r$. Then:*

$$\ell \cdot \Phi_\gamma(\ell) = \sum_{k=1}^{\ell} (-1)^{k-1} \sum_{\substack{\alpha+\beta=\gamma, \\ \alpha \neq (0,0,\ldots,0)}} c_\alpha(k)\, p_\alpha\, \Phi_\beta(\ell - k)$$

where in the second summation, α and β are in $\mathbb{N}^r$.

Proposition 1 and Corollary 1 replace the defining formula of the volume function, that implies explicitly the coordinate of the roots, with formulas in terms of the vector symmetric functions of the roots. Those can be computed easily from the equations in a variety of situations, as explained at the end of Section 3.

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Emmanuel Briand, Universidad de Sevilla

Mercedes Rosas, Universidad de Sevilla