On a matrix function interpolating between determinant and permanent

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Abstract

A matrix function, depending on a parameter t, and interpolating between the determinant and the permanent, is introduced. It is shown this function admits a simple expansion in terms of determinants and permanents of sub-matrices. This expansion is used to explain some formulas ocurring in the resolution of some systems of algebraic equations.

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1 A function interpolating between permanent and determinant

Given a permutation σ of $\{1, \ldots, r\}$, denote with $\mathcal{O}(\sigma)$ the set of its orbits, and define the following class function with parameter:

$$\gamma_t(\sigma) = \prod_{\omega \in \mathcal{O}(\sigma)} \left(1 - t^{|\omega|}\right).$$

Here $|\omega|$ is the cardinality of ω. One can now consider the following function, defined upon the square matrices of order r:

$$\Gamma_t(M) = \sum_{\sigma \in \mathfrak{S}_r} \gamma_t(\sigma) \prod_i m_{i,\sigma(i)}. \tag{1}$$

The object of this note is the following identity:

Theorem 1

$$\Gamma_t(M) = \sum (-t)^{|I|} \det M[I] \operatorname{per} M[I^*]. \tag{2}$$

where the sum is carried over the subsets I of $\{1, \ldots, r\}$.

The notation $M[I]$ refers to the submatrix of M obtained by keeping only the lines with index in I, and as well the columns with index in I; and I^* is the complement of I.

Proof : Expanding the product

$$\gamma_t(\sigma) = \prod_{\omega \in \mathcal{O}(\sigma)} (1 - t^{|\omega|}),$$

one gets

$$\sum_{\mathcal{U} \subset \mathcal{O}(\sigma)} (-t)^{\sum_{\omega \in \mathcal{U}} |\omega|}.$$

Associate to each subset $\mathcal{U}$ of $\mathcal{O}(\sigma)$ the union I of the orbits that are elements of $\mathcal{U}$. This way each subset stabilized by σ is obtained exactly once. The permutation σ induces a permutation σ_I of I and a permutation σ_{I^*} of its complement; the signature of σ_I is $(-1)^{|I|-|\mathcal{U}|}$. Thus the product is also equal to

$$\sum_{I \mid \sigma(I)=I} (-t)^{|I|} \, \varepsilon(\sigma_I).$$

Using this in the formula (1) where $\Gamma_t(M)$ is defined, it comes:

$$\sum_{\sigma \in \mathfrak{S}_r, I \mid \sigma(I)=I} (-t)^{|I|} \varepsilon(\sigma_I) \prod_i m_{i,\sigma(i)} \ ,$$

which decomposes into

$$\begin{aligned}\sum_{\sigma \in \mathfrak{S}_r, I \mid \sigma(I)=I} & (-t)^{|I|} \varepsilon(\sigma_I) \prod_{i \in I} m_{i,\sigma_I(i)} \prod_{i \in I^*} m_{i,\sigma_{I^*}(i)} \\ &= \sum_I (-t)^{|I|} \det M[I] \operatorname{per} M[I^*]\end{aligned}$$

And this is the announced result. ■

The function Γ_t interpolates between the permanent and the determinant because:

- when $t = 0$, that is: when considering the constant term, one obtains the permanent.
- when $t \to -\infty$, that is: when considering the leading coefficient, that of t^n, one obtains the determinant.

Another interesting specialization is obtained with $t = 1$; it gives an identity due to Muir ([1]):

$$0 = \sum_I (-1)^{|I|} \det M[I] \operatorname{per} M[I^*].$$

Yet another specialization, $t = 1/2$, appears in the study of certain systems of algebraic equations, where the identity (2) happens to be useful. This is what we expose in the next part.

2 An application to some systems of polynomial equations

Let a system of r quadratic equations $F_1 = \ldots = F_r = 0$ in r unknowns $X_1, \ldots, X_r$, with the following particular shape:

$$F_i = X_i^2 + \sum_{j=1}^{r} u_{i,j} X_j + c_i.$$

Such a system has always finitely many roots, namely 2^r if each root is counted as many times as indicated by its multiplicity. We'll denote with U the matrix of the u_{ij}. One wants to calculate the multisymmetric power sums of the multiset of the roots of this system[1], denote them with p_α for $\alpha \in \mathbb{N}^r$. This means that p_α is the sum of the evaluations of the monomial $X_1^{\alpha_1} \cdots X_r^{\alpha_r}$ at the roots (repeated when there are multiplicities). The power sums which may be difficult to obtain are the first ones: the p_α with α with

[1]Such analogues of the symmetric functions where introduced by several authors at the end of the nineteenth century. See [2] for more references and specially connections to systems of polynomial equations with finitely many solutions.

$0-1$ coordinates[2]). These first power sums may be computed using the *identities of Aizenberg and Kytmanov* ([3])[3]:

$$\sum_{\alpha\in\mathbb{N}^r} \frac{p_\alpha}{X^\alpha} = \Omega\,\mathcal{S}\,\frac{X_1\cdots X_r\,\mathrm{Jac}}{F_1\cdots F_r} \tag{3}$$

In the right-hand side of the formula, Jac is the Jacobian of $F_1,\ldots,F_r$; the $\mathcal{S}$ symbol means that a series expansion, following the decreasing total degree, is applied to the rational fraction (this is possible because the denominator has exactly one term of maximal total degree, that is: $X_1^2\cdots X_r^2$), and next that all the Laurent monomials *with numerator* are removed from this expansion (Ω symbol).

Applying this to the first examples of the considered systems, one finds that

- for $r=1$,
$$p_1 = -u_{11},$$
- for $r=2$,
$$p_{11} = u_{11}u_{22} + 3u_{12}u_{21},$$
- for $r=3$,
$$\begin{aligned} p_{111} = -u_{11}u_{22}u_{33} - 3u_{11}u_{23}u_{32} - 3u_{13}u_{22}u_{33} - 3u_{12}u_{21}u_{33} \\ - 7u_{12}u_{23}u_{31} - 7u_{13}u_{21}u_{32}. \end{aligned}$$

These observations, and a few next ones, lead to conjecture that

$$p_{11\cdots1} = (-1)^r \sum_{\sigma\in\mathfrak{S}_r} \prod_{\omega\in\mathcal{O}(\sigma)} (2^{|\omega|}-1) \prod_i u_{i,\sigma(i)}. \tag{4}$$

Making use of the identity (2), we'll prove the above formula, and the more general one:

Theorem 2

$$p_A = (-1)^{|A|}2^{n-|A|} \sum_{\sigma\in\mathfrak{S}_A} \prod_{\omega\in\mathcal{O}(\sigma)} (2^{|\omega|}-1) \prod_{i\in A} u_{i,\sigma(i)}. \tag{5}$$

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[2] The other power sums are easily deduced from the first ones. For instance, the first equation of the system yields the inductive formula

$$p_{(\alpha_1+2),\alpha_2,\ldots,\alpha_r} + \sum_j u_{1,j}\; p_{\alpha_1,\ldots,(1+\alpha_i),\ldots,\alpha_r} + c_1\; p_\alpha = 0$$

[3] Such identities were first written by Jacobi, in [4].

Here was made the following abuse of notation: given a subset A of $\{1,\ldots,n\}$, it was also denoted with A the vector whose i-th coordinate is 1 if $i \in A$, 0 if $i \notin A$.

Proof : It is convenient to re-write the quotient under the form:

$$\frac{\mathrm{Jac}/(X_1\cdots X_r)}{(F_1/X_1^2)\cdots(F_r/X_r^2)}$$

so that each of its term lies in the sub-algebra of Laurent monomials generated by the $1/X_i$ and the X_i/X_j^2; that is on the one hand

$$\frac{\mathrm{Jac}}{X_1\cdots X_r} = \sum_I 2^{n-|I|}\,\frac{\det U[I]}{X^I},$$

and in the other

$$\frac{F_i}{X_i^2} = 1 + \sum_j u_{i,j}\frac{X_j}{X_i^2} + \frac{c_i}{X_i^2}.$$

Now we may work *modulo* the ideal generated by the $1/X_i^2$ and the $(X_j/X_i^2)^2$, since we are only interested in the monomials with shape $1/X^A$, which all lie outside this ideal.

One has then

$$\frac{F_i}{X_i^2} \equiv 1 + \sum_j u_{i,j}\frac{X_j}{X_i^2} \equiv \prod_j \left(1 + u_{i,j}\frac{X_j}{X_i^2}\right)$$

which has as inverse

$$\prod_j \left(1 - u_{i,j}\frac{X_j}{X_i^2}\right).$$

Thus it remains to compute the terms in $1/X^A$ in the expression:

$$\sum_I 2^{n-|I|}\,\frac{\det U[I]}{X^I}\prod_{i,j}\left(1 - u_{i,j}\frac{X_j}{X_i^2}\right)$$

and in this aim, it is now sufficient to work *modulo* the module over the polynomials in $1/X_1,\ldots,1/X_r$ generated by the X_j/X_i^2 for $i \neq j$. One has then:

$$\prod_{i,j}\left(1 - u_{i,j}\frac{X_j}{X_i^2}\right) \equiv \sum_I (-1)^{|I|}\frac{\mathrm{per}\,U[I]}{X^I}$$

because the $X^J/(X^I)^2$ that appear in the expansion of the product are in the sub-module unless $I = J$. Finally the coefficient of $1/X^A$ is given by

$$\sum_{I \subset A} 2^{n-|I|} \det U[I] (-1)^{|A|-|I|} \operatorname{per} U[A \setminus I]$$

and one recognizes in it

$$(-1)^{|A|} 2^n \Gamma_{1/2}(U[A]),$$

which re-writes easily as formula (5). ■

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