The configurations of two projective conics

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Quadric day

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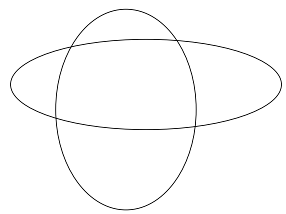
0 – Objectives

1. Classify the pairs of distinct, non-empty, proper (= non-degenerate) conics of the projective plane.
2. Find the general equations, inequations, inequalities characterizing the classes.
3. develop methods apliable for the same problem for of *quadrics*.

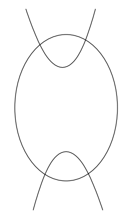
0 – Plan

1. The results: classes and their description.
2. The methods for classification.
3. The production of the equations
4. By-product: characterization of the classes by means of the *behaviour of the signature function.*

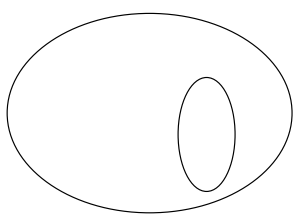
1 – Results: generic classes



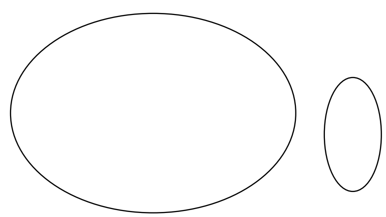
IN



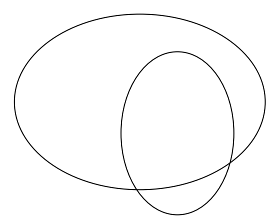
IS



$IaN(*)$

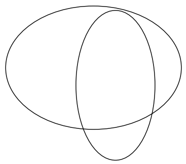


IaS

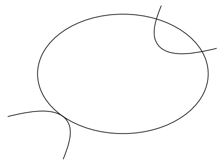


IbN

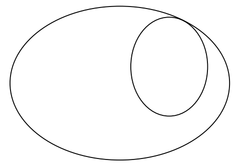
1 – Results: non-generic classes



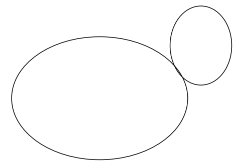
$IIN(*)$



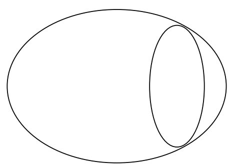
IIS



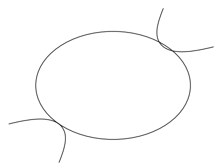
$IIaN(*)$



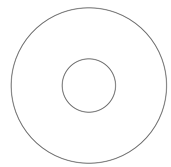
$IIaS$



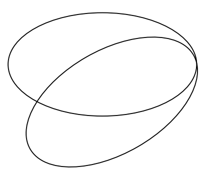
$IIIN(*)$



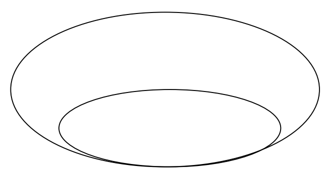
$IIIS$



$IIIaN(*)$



IVN



$VN(*)$

1 – D

escriptions

$$
\begin{aligned}
I: &\quad \mathrm{Disc}(\Phi) < 0 \wedge p_2 < 0 \wedge \Phi_{30} A_1 > 0 \\
Ia: &\quad \mathrm{Disc}(\Phi) < 0 \wedge [p_2 > 0 \vee \Phi_{30} A_1 < 0 \vee [A_1 = 0 \wedge p_2 = 0]] \\
Ib: &\quad \mathrm{Disc}(\Phi) > 0 \\
II: &\quad \mathrm{Disc}(\Phi) = 0 \wedge H \neq 0 \wedge G \neq 0 \wedge p_2 < 0 \wedge \Phi_{30} A_1 > 0 \\
IIa: &\quad \mathrm{Disc}(\Phi) = 0 \wedge H \neq 0 \wedge G \neq 0 \wedge [p_2 = 0 \vee \Phi_{30} A_1 < 0] \\
III: &\quad \mathrm{Disc}(\Phi) = 0 \wedge H \neq 0 \wedge G = 0 \wedge p_2 < 0 \\
IIIa: &\quad \mathrm{Disc}(\Phi) = 0 \wedge H \neq 0 \wedge G = 0 \wedge p_2 > 0 \\
IV: &\quad H = 0 \wedge G \neq 0; \qquad V: \quad H = 0 \wedge G = 0.
\end{aligned}
$$

And N *versus* S is determined by:

$$N \quad \Leftrightarrow \quad \Phi_{30}\Phi_{12} > 0 \wedge \Phi_{03}\Phi_{21} > 0.$$

1 – Formulas

If f and g are two quadratic forms defining the conics,

$$\Phi(t,u) = \mathrm{Disc}(tf+g) = \Phi_{30}t^3 + \Phi_{21}t^2u + \Phi_{12}tu^2 + \Phi_{03}u^3$$

is the characteristic form associated to (f,g). Appear also:

- Its discriminant $\mathrm{Disc}(\Phi)$.
- Its hessian H.
- The triangle covariant G: a product of three linear forms whose zero locus are the sides of a triangle associated to the two conics.
- A_1 and p_2: subresultants for ...

2 – Criteria of classification

The objective will be classification *modulo* ambiant isotopy = continuous family of homeomorphisms of the ambiant projective space, starting from **Id**.

But

the tool will be classification *modulo* rigid isotopy = continuous deformation of the equations of the conics not changing the nature and nof the singularities of the pair of conics.

Here the singularities are the intersection points, their nature is their analytic type.

2 – Intersections of two conics

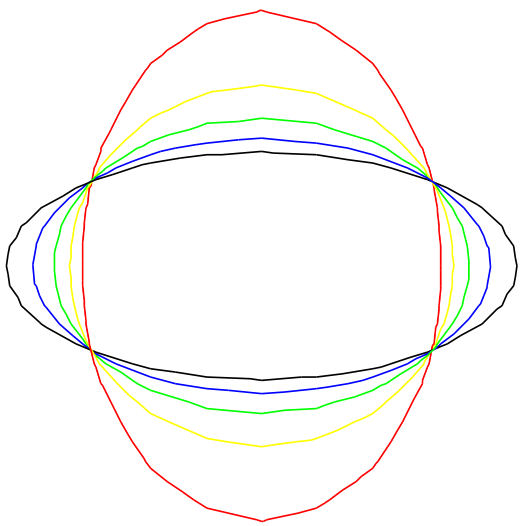
The analytic type of an intersection point of two conics is totally determined by its multiplicity and its nature (*real or imaginary*).

The space of couples of distinct, non-empty, proper conics is partitionned following the numbers of real and imaginary multiplicity (ex: one real double point and two imaginary simple points).

A rigid isotopy is a continuous path in one of these subsets.

2 – Pencils of conics

Two distinct conics generate a line in the space of conics; it is called a pencil of conics.



The real and imaginary intersection points, and their multiplicities, depend only on the pencil, not on the pair of conics on it.

2 – Orbits of pencils of conics

The group of projective transformations acts on the pencils. Levy, Degtyarev : there are 9 orbits for non-degenrate pencils (*i.-e.*) pencils not totally made of degenerate quadrics).

And they are characterized by the numbers of real and imaginary base poinst of each multiplicity.

Orbit	I	Ia	Ib	II	IIa	III	$IIIa$	IV	V
real points	1111	–	11	211	2	22	–	31	4
imaginary points	–	1111	11	–	11	–	22	–	–

2 – Rigid isotopies and orbits of pencils

There is a projection:

$$\text{Pair of conics} \longmapsto \text{the pencil they generate}$$

The subsets of pairs of conics whose number of real and imaginary intersection points of each multiplicty are prescribed are exactly the inverse images of the orbits.

Specially, they are submanifolds. The classes of rigid isotopy are exactly their connected components.

2 – Rigid isotopy and ambiant isotopy

Rigid isotopy = deformation of the equations.

Ambiant isotopy = deformation of the plane.

Theorem. *If* (C_1, C_2) *and* (C'_1, C'_2) *are connected by a rigid isotopy, then they are alos connected by an ambiant isotopy.*

To prove this theorem.

2 – Decomposition of a rigid isotopy

Consider a rigid isotopy $(C_1(t), C_2(t))$ connecting $(C_1(0), C_2(0))$ to $(C_1(1), C_2(1))$.

This is a path in one strata of the space of couples of conics; it projects to a path drawn in some projective orbit in the space of pencils. This latter lifts to a path from θ_t, from **Id**, in the space of projective transformations:

$$\theta_t \langle C_1(0), C_2(0) \rangle = \langle C_1(t), C_2(t) \rangle.$$

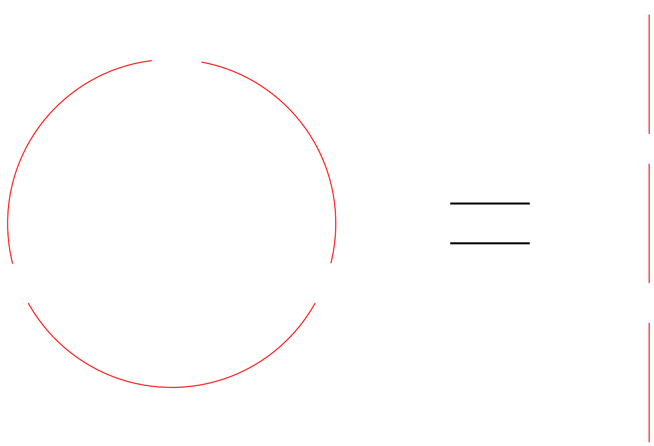
Consider $\theta_t^{-1}(C_1(t), C_2(t))$, this path in the space of couples of conics is drawn in the pencil: $\langle C_1, C_2 \rangle$.

Find an ambiant isotopy β_t connecting as well $(C_1(0), C_2(0))$ to $\theta_1^{-1}(C_1(1), C_2(1))$. Then $\beta_t \circ \theta_t$ will connect $(C_1(0), C_2(0))$ to $(C_1(1), C_2(1))$.

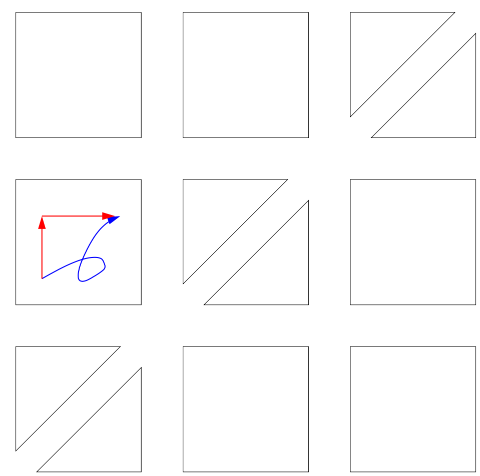
2 – Rigid isotopy drawn in a pencil

If two couples of conics are connected by a rigid isotopy drawn in a pencil, they are also connected by special rigid isotopies of the following kind: $([f+tg],[g])$ and $([f],[g+tf])$.

$\mathbb{RP}^1 \setminus \mathrm{Deg}$



$(\mathbb{RP}^1 \setminus \mathrm{Deg}) \times (\mathbb{RP}^1 \setminus \mathrm{Deg}) \setminus \mathrm{Diag}$

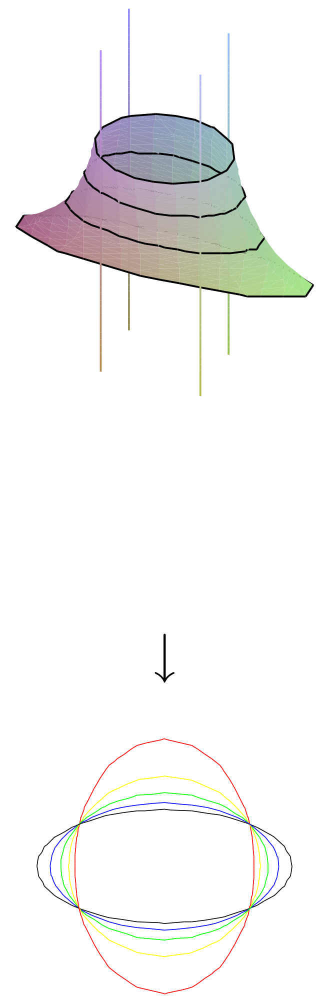


For small t, the rigid isotopy $([f+tg],[g])$ is realized by an ambiant isotopy.

Proof: build first the ambiant isotopy on the *blow-up*

$$((x:y:z;f(x,y,z):g(x,y,z)))$$

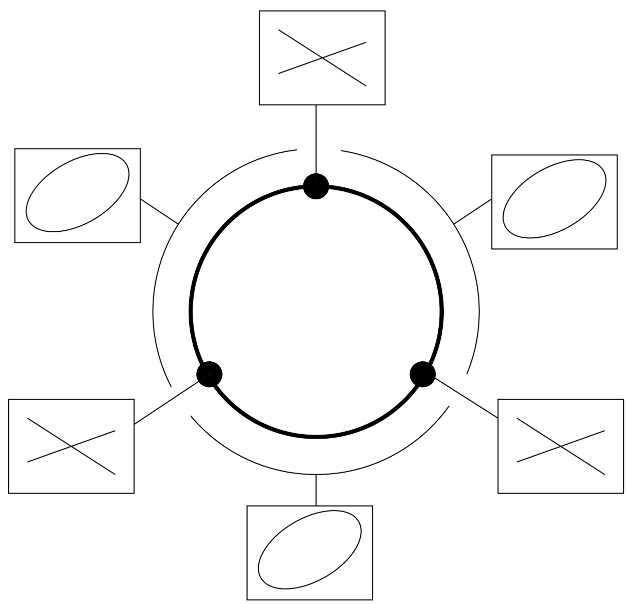
in which the conics appear as separated.



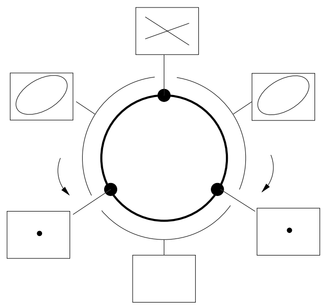
2 – Portrait of a pencil

To classify the couples of conics, we'll use the portraits of the pencils.

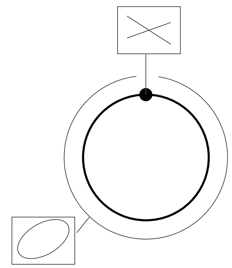
For I:



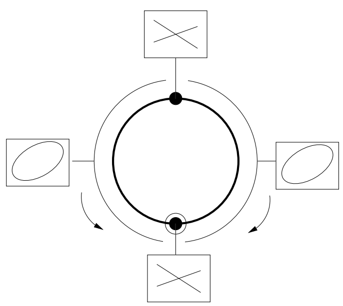
2 – Portraits of the pencils (2 of 3)



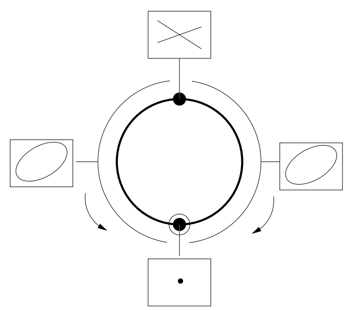
Ia



Ib

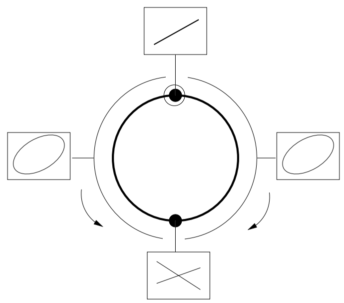


II

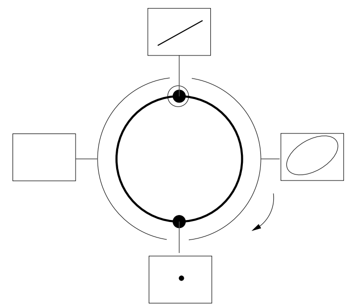


IIa

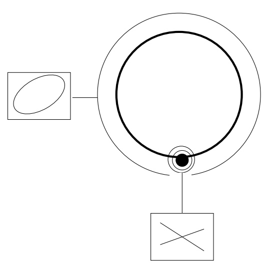
2 – Portraits of the pencils (3 of 3)



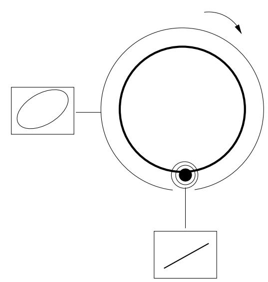
III



$IIIa$



IV



V

2 – Classification for *pairs*

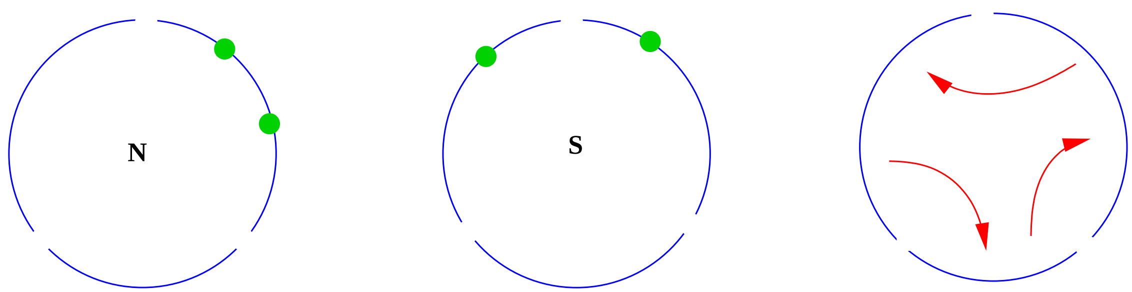
To produce a system of representants for the clases *modulo* rigid isotopy and interchange of the two conics:

1. for each of the 9 orbits of pencils, choose a pencil-representant.

2. for each of the pencil-representants, choose:

   - a pair of conics on each arc.

   - a pair of conics for each pair of arcs, one conic on each of the two arcs.



2 – Classification for pairs (contd.)

For each orbit of pencil, we get one (N) or two (N and S) representants of classes of pairs.

By simple topological considerations, we check they are not equivalent.

2 – **Classification for *couples***

Each class of pairs:

- either is a class of couples.For the representant $\{[f],[g]\}$: $([f],[g]) \cong ([g],[f])$
- either splits into two classes of couples: $([f],[g]) \not\cong ([g],[f])$ $(*)$.

In the first case, it is always possible to exhibate a projective tranformation realizing the equivalence.

In the second case, one sees that always one of the conics lies inside the other.

IN	$3x^2 - 2y^2 - z^2$	$3x^2 - y^2 - 2z^2$
IS	$3x^2 - 2y^2 - z^2$	$x^2 - 2y^2 + z^2$
$IaN(*)$	$x^2 + y^2 + z^2 + 3xz$	$x^2 + y^2 + z^2 + 4xz$
IaS	$x^2 + y^2 + z^2 + 3xz$	$x^2 + y^2 + z^2 - 3xz$
IbN	$x^2 + y^2 - z^2 + xz$	$x^2 + y^2 - z^2 - xz$
$IIN(*)$	$yz + xy - xz$	$yz + 2xy - 2xz$
IIS	$yz + xy - xz$	$yz - xy + xz$
$IIaN(*)$	$y^2 + z^2 + xz$	$y^2 + z^2 + 2xz$
$IIaS$	$y^2 + z^2 + xz$	$y^2 + z^2 - xz$
$IIIN(*)$	$xz+^2$	$xz + 2y^2$
$IIIS$	$xz + y^2$	$xz - y^2$
$IIIaN(*)$	$x^2 + y^2 - z^2$	$x^2 + y^2 - 2z^2$
IVN	$xz - y^2 + xy$	$xz - y^2 - 2xy$
$VN(*)$	$xz - y^2 - x^2$	$xz - y^2 + x^2$

2 – Ambiant isotopy classes

They're deduced from the rigid isotopy classes.

$$\begin{array}{cccc} IN & IS & IaS & IbN \cup IVN \\ IIN & IIaS & IIIS & (IaN \cup IIIaN)(*) \\ IIS(*) & (IIaN \cup VN)(*) & IIIN(*) & IIIaN(*) \end{array}$$

3 – Equations, *etc*

We want to characterize the rigid isotopy classes by equations, inequations and inequalities.

1. Characterize the strata, inverse images of the orbits of pencils.
2. For each strata, characterize the relative position of the pair $([f],[g])$ and the degenerate conics of its pencil (when relevant).
3. characterize the relative position of $[f]$ and $[g]$ and the degenerate conics to decide which is inside the other (when relevant).

3 Characterization of the strata

The strata are characterized:

- using invariants and covariants of couples of ternary quadratic forms (`Glenn`):

$$I \quad Ia \quad | \quad Ib \quad | \quad II \quad IIa \quad | \quad III \quad IIIa \quad | \quad IV \quad | \quad V.$$

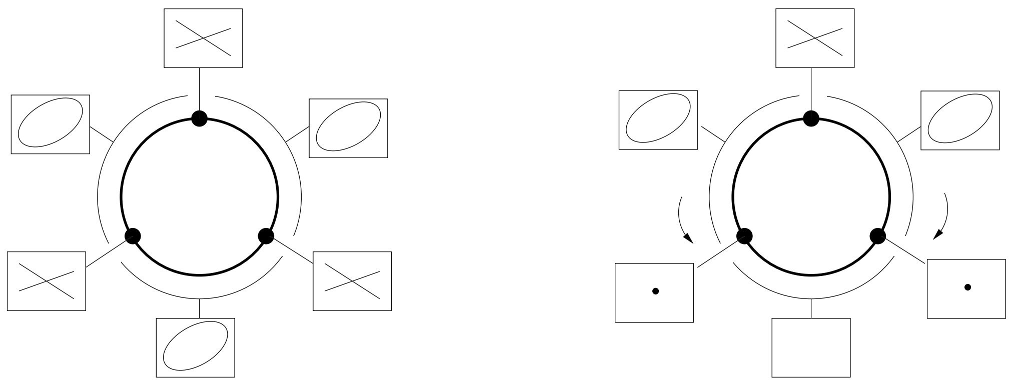
- and using the specific features of the pencils of each orbit.

3 – **Ex:** I *vs.* Ia

The characteristic polynomial of $f + tg$:

$$\chi(X) = X^3 - \nu(t)X^2 + \psi(t)X - \phi(t).$$

portraits:

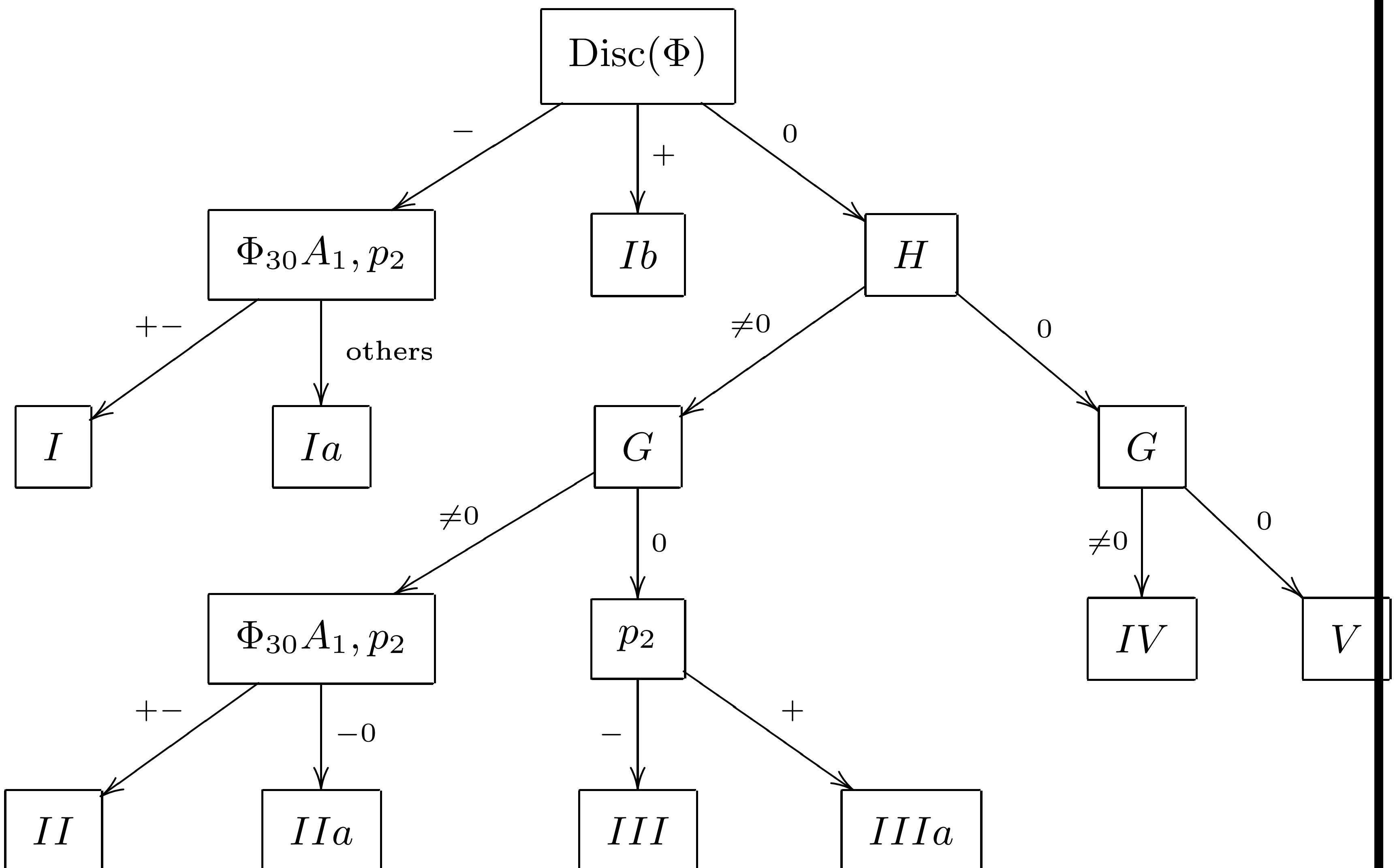


eigenvalues: + -,+ -,+ -   + -, + + o - -, + + o - -

ψ on the roots of ϕ - - -   +++

It corresponds to inequalities for the subresultants of $\psi \cdot \phi'$ and ϕ.

(1.) Characterization of the strata



(2). Characterization of N *vs.* S

Class	N	S
Pencil	same arc	distinct arcs
$\phi(t) = \mathrm{Disc}(tf + g)$	3 roots of the same sign	other cases

From Descartes' rule of signs one concludes. If

$$\phi(t) = \Phi_{30}t^3 + \Phi_{21}t^2 + \Phi_{12}t + \Phi_{03}$$

Then the class N is characterized by:

$$\Phi_{30}\Phi_{12} > 0 \wedge \Phi_{03}\Phi_{21} > 0.$$

(3.) Which is inside the other ?

Invariants, covariants and subresultants give the inequalities answering in any case.

4 – Aplication: decide the relative position of two conics [with equations] depending on parameters

1. Specielize the general equations, inequations, inequalities to the particular case.
2. Simplify the cumbersome formulas obtained (using Christopher Brown's SLFQ).

4 – Example: two ellipsoids in $\mathbb{R}^3$

$$x^2 + y^2 + z^2 - 25 = 0,$$
$$\frac{(x-6)^2}{9} + \frac{y^2}{4} + \frac{z^2}{16} - 1 = 0.$$

Homogenize these polynomials estos in x, y with t:

$$\begin{aligned} f &= \quad x^2 + y^2 + t^2(z^2 - 25), \\ g &= \quad \tfrac{(x-6t)^2}{9} + \tfrac{y^2}{4} + t^2\left(\tfrac{z^2}{16} - 1\right). \end{aligned}$$

These are conics depending on the parameter z. One gets the discriminant:

$$h = 49z^4 + 2516z^2 - 229376.$$

After simplification, one gets:

- IaS cuando $h > 0$, *i.-e.* $-4 < z < -z_0$ o $z_0 < z < 4$.
- $IIaS$ cuando $h = 0$, *i.-e.* $z = \pm z_0$.
- IbN cuando $h < 0$, *i.-e.* $-z_0 < z < z_0$.

4 – Aplication: the signature function

Wang and Krasauskas: the signature function characterizes the generic configurations of two conics.

Let us show that: the signature functioncharacterizes the (generic and non-generic) configurations of two conics (nearly totally).

Signature of $\pm g \pm tf$ (1 of 2)

IN		$\widehat{21}(11)12(11)21(11)12$
IS		$21(11)\widehat{12}(11)21(11)12$
IaN	$f < g$	$\widehat{21}(11)12(02)03(02)12$
IaN	$f > g$	$21(11)12(02)03(02)\widehat{12}$
IaS		$21(11)\widehat{12}(02)03(02)12$
IbN		$\widehat{21}(11)12$
IIN	$f < g$	$21((11))12(11)\widehat{12}$
IIN	$f > g$	$\widehat{21}(11)12(11)12$
IIS		$21((11))\widehat{12}(11)12$
$IIaN$	$f < g$	$21((20))21(11)\widehat{12}$

Signature of $\pm g \pm tf$ (2 of 2)

$IIaN$	$f > g$	$\widehat{21}((20))21(11)12$
$IIaS$		$21((20))\widehat{21}(11)12$
$IIIN$	$f < g$	$\widehat{21}((10))21(11)12$
$IIIN$	$f > g$	$21((10))21(11)\widehat{12}$
$IIIS$		$21((10))\widehat{21}(11)12$
$IIIaN$	$f < g$	$\widehat{21}((01))03(02)12$
$IIIaN$	$f > g$	$21((01))03(02)\widehat{12}$
IVN		$\widehat{21}(((11)))12$
VN	$f < g$	$\widehat{21}(((10)))12$
VN	$f > g$	$\widehat{21}(((10)))12$

∞ – **Perspectives**

Classification for quadric surfaces, using Uhlig's work.

The End